Math 116 — First Midterm — October 1, 2024

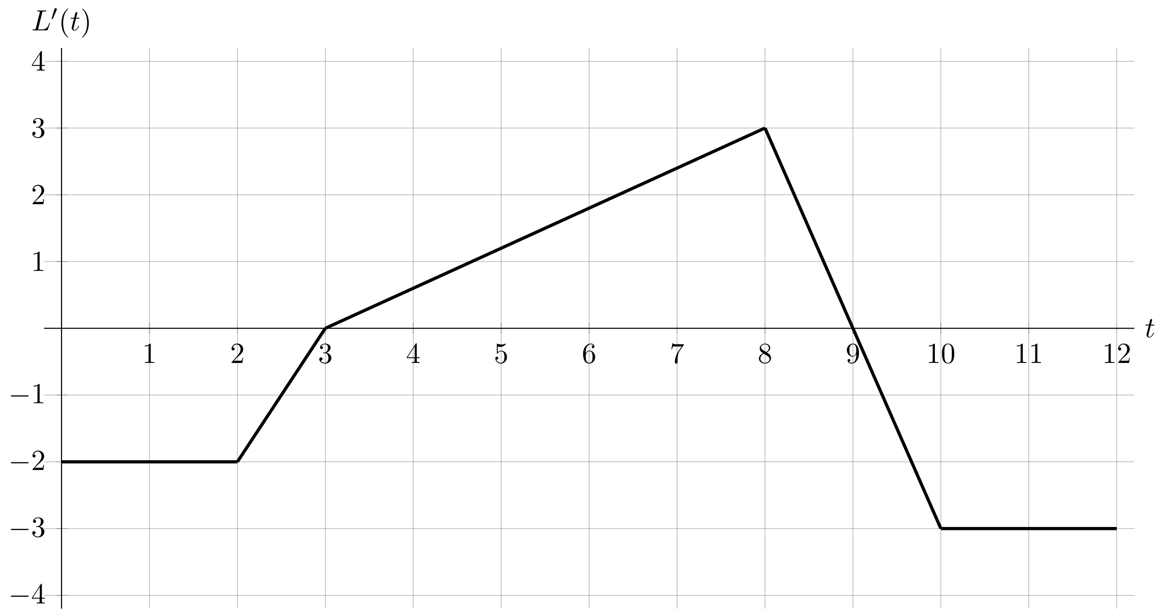
EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 14 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	5	
2	14	
3	15	
4	8	
5	15	

Problem	Points	Score
6	7	
7	10	
8	8	
9	8	
10	10	
Total	100	

1. [5 points] Emily runs a lemonade stand. Her cumulative net profit fluctuates throughout the year. The function $L(t)$ represents the cumulative net profit of Emily's lemonade stand, in dollars, t months after January 1, 2024. Below is a graph of $L'(t)$, the **derivative** of $L(t)$.



For each part below, circle the **one** best option.

a. [2 points] Based on the graph of $L'(t)$, on what point in 2024 will the cumulative net profits of Emily's lemonade stand be largest?

i. $t = 0$ | iii. $t = 8$ | v. $t = 12$

ii. $t = 3$ | iv. $\boxed{t = 9}$ | vi. NONE OF THESE

b. [2 points] There is a chocolate cake at a nearby store that Emily really wants to buy, but she wants the cumulative net profits of her lemonade stand to be at least \$10 before she buys the chocolate cake. What is the smallest that $L(0)$ could be in order for her to be able to buy the chocolate cake at some point in 2024?

i. \$0 ii. \$2 iii. \$4 iv. $\boxed{\$6}$ v. \$8 vi. \$10

c. [1 point] Based on the graph of $L'(t)$, Emily tries to make a graph of $L(t)$ by assuming that the cumulative net profits of her lemonade stand are P_0 dollars on January 1, 2024. Later she discovers that the cumulative net profits on January 1, 2024 were instead $P_0 + 3$ dollars. How should Emily change her graph of $L(t)$ to reflect this discovery?

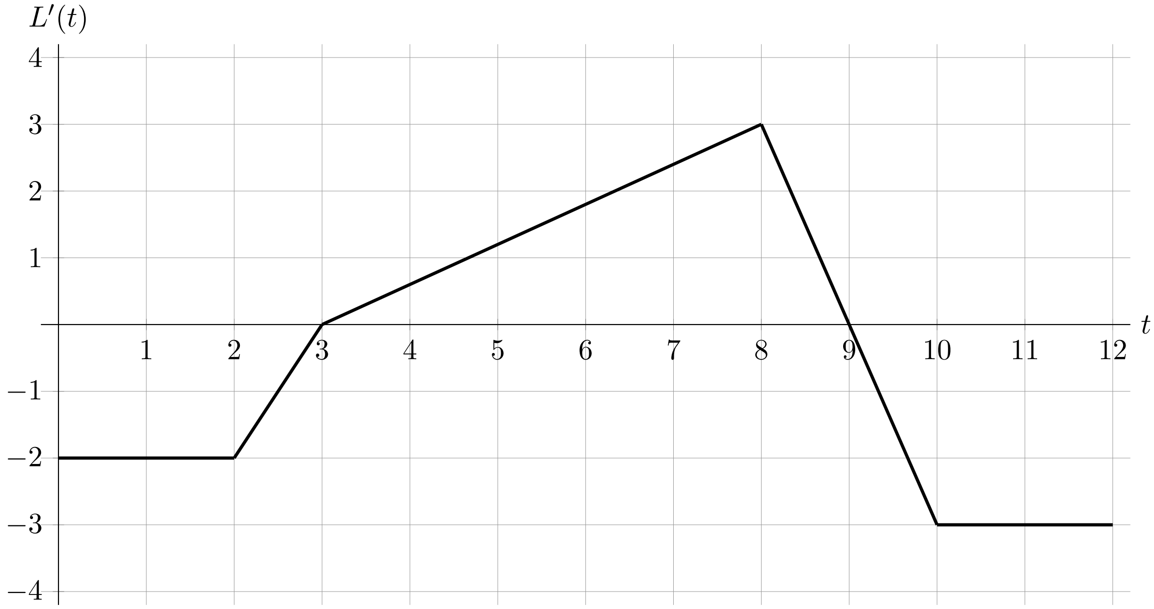
i. $\boxed{\text{Shift } L(t) \text{ up by } 3}$

iii. Shift $L(t)$ up by $P_0 + 3$

ii. Shift $L(t)$ down by 3

iv. Shift $L(t)$ down by $P_0 + 3$.

2. [14 points] The graph of $L'(t)$ from the previous page has been replicated here.

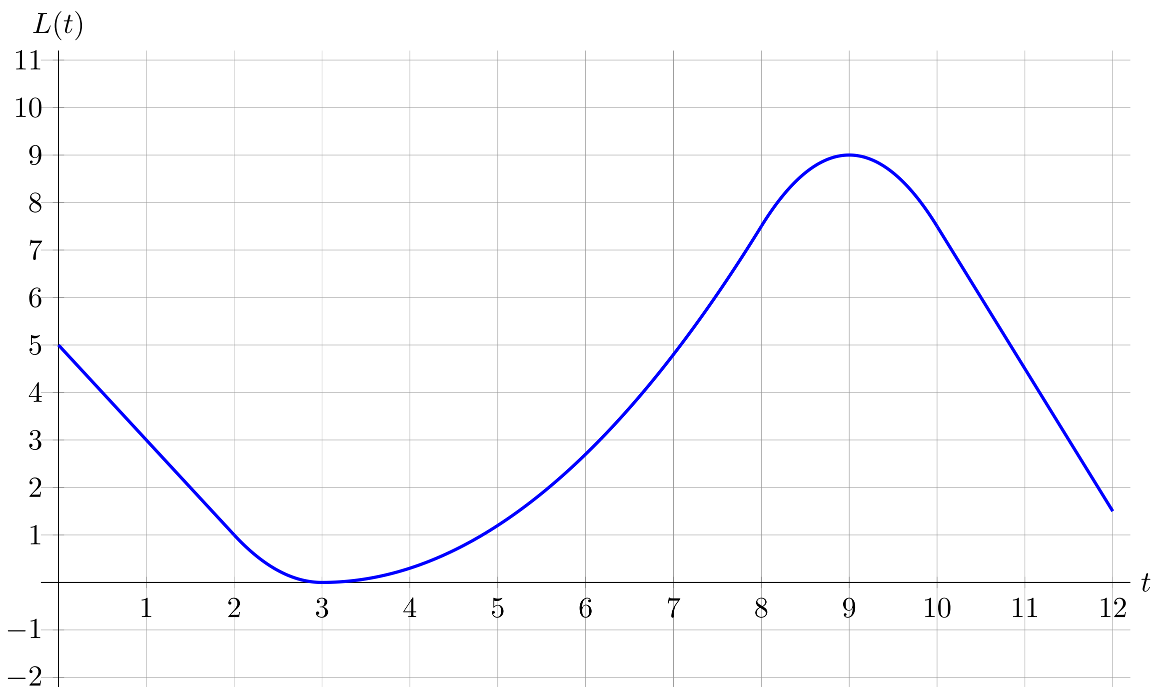


Now assume that the cumulative net profit of Emily's lemonade stand is \$5 at the beginning of January 1, 2024.

a. [4 points] Fill in the following table of values for $L(t)$.

t	0	2	3	8	9	10	12
$L(t)$	5	1	0	7.5	9	7.5	1.5

b. [10 points] Sketch a graph of the cumulative net profits of Emily's business over 2024. Make sure to clearly label the values of your graph at $t = 2, 3, 8, 9, 10, 12$ and indicate where your graph is concave up, concave down, or linear.



3. [15 points] Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is always positive, and is also differentiable. Some values of $f(x)$ are given in the table below:

x	1	2	3	4	5	6
$f(x)$	1	3	6	13	20	22

Compute the exact value of the following integrals. If there is not enough information provided to determine the value of the integral, write "NEI" and clearly indicate why. Show all of your work.

a. [5 points] $\displaystyle\int_1^2 f'(3x)e^{f(3x)+3}\,dx$.

Solution: We use the substitution,

$$u = f(3x) + 3 \qquad du = 3f'(3x)$$

Then

$$\int_1^2 f'(3x)e^{f(3x)+3}\,dx = \frac{1}{3}\int_9^{25} e^u\,du = \frac{1}{3}\left(e^{25} - e^9\right)$$

Answer: $\frac{1}{3}\left(e^{25} - e^9\right)$

b. [5 points] $\displaystyle\int_1^6 f'(x)\left(1 + \ln(f(x))\right)\,dx$.

Solution: We set

$$u = f(x), \qquad du = f'(x)dx$$
$$v = 1 + \ln(f(x)), \qquad dv = \frac{f'(x)}{f(x)}dx$$

Then, integrating by parts,

$$\begin{aligned}\int_1^6 f'(x)\left(1 + \ln(f(x))\right)\,dx &= f(x)(1 + \ln(f(x)))\Big|_1^6 - \int_1^6 f'(x)\,dx \\ &= f(x)(1 + \ln(f(x)))\Big|_1^6 - f(x)\Big|_1^6 \\ &= f(x)\ln(f(x))\Big|_1^6 \\ &= 22\ln(22) - 1\ln(1) \\ &= 22\ln(22)\end{aligned}$$

Answer: $22\ln(22)$

c. [5 points] $\displaystyle\int_2^5 \frac{f'(x) + \frac{1}{x}}{(f(x) + \ln(x))^2}\,dx$.

Solution: Let

$$u = f(x) + \ln(x) \qquad du = \left(f'(x) + \frac{1}{x}\right) dx$$

Then

$$\begin{aligned}
\int_2^5 \frac{f'(x) + \frac{1}{x}}{(f(x) + \ln(x))^2}\, dx &= \int_{3+\ln(2)}^{20+\ln(5)} \frac{1}{u^2}\, du \\
&= -\frac{1}{u}\Big|_{3+\ln(2)}^{20+\ln(5)} \\
&= -\frac{1}{20+\ln(5)} + \frac{1}{3+\ln(2)}
\end{aligned}$$

Answer: $-\dfrac{1}{20+\ln(5)} + \dfrac{1}{3+\ln(2)}$

4. [8 points] Consider the following function:

$$F(x) = \int_1^{\ln x} \frac{\cos^2(t)}{t}\, dt.$$

a. [2 points] Find a value of a such that $F(a) = 0$. Show your work.

Solution: $F(a) = 0$ when the upper and lower bounds are equal, in other words when $\ln(a) = 1$. This happens when $a = e$.

Answer: $a =$ e

b. [3 points] Calculate $F'(x)$.

Solution: We can use the chain rule to find $F'(x)$:

$$F'(x) = \frac{\cos^2(\ln(x))}{\ln(x)} \frac{d}{dx}(\ln x) = \frac{\cos^2(\ln(x))}{\ln(x)} \frac{1}{x} = \frac{\cos^2(\ln(x))}{x\ln(x)}.$$

Answer: $F'(x) =$ $\dfrac{\cos^2(\ln(x))}{x\ln(x)}$

c. [3 points] Find a function $f(t)$ and constants a and C so that we may rewrite $F(x)$ in the form $\displaystyle\int_a^x f(t)\, dt + C$. There may be more than one correct answer.

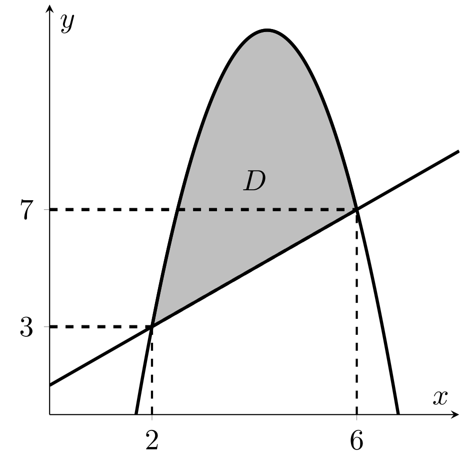
Solution: From our earlier work, we know that $F(x)$ is an antiderivative of $F'(x) = \dfrac{\cos^2(\ln(x))}{x\ln(x)}$ which satisfies $F(e) = 0$. Using the Second Fundamental Theorem of Calculus, we see that we may express:

$$F(x) = \int_e^x \frac{\cos^2(\ln(t))}{t\ln(t)}\, dt + 0$$

$f(t) =$ $\dfrac{\cos^2(\ln(t))}{t\ln(t)}$ $a =$ e $C =$ 0

5. [15 points]

Katydyd Delemma owns a donut shop. She likes to experiment with different shapes for her donuts. Katydyd decides that she'd like to make new donuts using the region D, which is the region in the first quadrant bounded by the curves $y = x + 1$ and $y = -2x^2 + 17x - 23$, as shaded in the figure to the right. The two curves intersect at the points $(2, 3)$ and $(6, 7)$.



a. [5 points] Write an expression involving one or more integrals for the volume of the donut generated by revolving the region D about the y-axis. **Do not** evaluate any integrals in your expression.

Solution: We use vertical slices, which gives rise to shells. For this region, x ranges between 2 and 6, so we get

$$\int_2^6 2\pi x\left((-2x^2+17x-23)-(x+1)\right)\,dx = \int_2^6 2\pi x\left(-2x^2+16x-24\right)\,dx$$

$$= -4\pi\int_2^6 x\,(x-2)\,(x-6)\,dx$$

Answer: $\int_2^6 2\pi x\left(-2x^2+16x-24\right)\,dx$

b. [5 points] Write another expression involving one or more integrals for the volume of the solid generated by revolving the region about the line $y = -1$. **Do not** evaluate any integrals in your expression.

Solution: We use vertical slices, which gives rise to washers. For this region, x ranges between 2 and 6, so we get

$$\int_2^6 \pi\left((-2x^2+17x-23-(-1))^2-(x+1-(-1))^2\right)\,dx$$

$$= \int_2^6 \pi\left((-2x^2+17x-22)^2-(x+2)^2\right)\,dx$$

$$= \int_2^6 \pi\left(4x^4-68x^3+376x^2-752x+480\right)\,dx$$

Answer: $\int_2^6 \pi\left((-2x^2+17x-22)^2-(x+2)^2\right)\,dx$

c. [5 points] Katydyd decides that she doesn't want to deal with all the oddly-shaped donut holes, so she decides to make little cakes instead. She wants the base to be the region D pictured above, with square cross-sections perpendicular to the x-axis. Find an expression involving one or more integrals for the volume of Katydyd's cakes. **Do not** evaluate any integrals in your expression.

Solution: The volume of a slice of small width Δx is approximately

$$((-2x^2 + 17x - 23) - (x + 1))^2 \Delta x = (-2x^2 + 16x - 24)\Delta x$$

Since x ranges between 2 and 6 in this region, the total volume of one cake is

$$\int_2^6 ((-2x^2 + 17x - 23) - (x + 1))^2 \, dx = \int_2^6 (-2x^2 + 16x - 24)^2 \, dx$$
$$= \int_2^6 4\left((x - 2)(x - 6)\right)^2 \, dx$$

Answer: $\int_2^6 (-2x^2 + 16x - 24)^2 \, dx$

6. [7 points] Use the partial fraction decomposition

$$\frac{x^2+13x+10}{(7-x)(x^2+1)} = \frac{3}{7-x} + \frac{2x+1}{x^2+1}$$

to evaluate the following indefinite integral, showing all of your work.

$$\int \frac{x^2+13x+10}{(7-x)(x^2+1)}\,dx$$

Solution:

$$\begin{aligned}\int \frac{x^2+13x+10}{(7-x)(x^2+1)}\,dx &= \int \frac{3}{7-x} + \frac{2x+1}{x^2+1}\,dx \\ &= \int \frac{3}{7-x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= -3\ln|7-x| + \ln|x^2+1| + \arctan x + C\end{aligned}$$

Answer: $-3\ln|7-x| + \ln|x^2+1| + \arctan x + C$

7. [10 points] On her latest hiking expedition, Emily has scaled up a cliff face with a height of 30 feet, and is now using her rope to pull up her camping supplies after her. The supplies are initially on the ground, and are pulled directly upward by the rope. Assume that the rope initially extends from the top of the cliff to the ground in a straight vertical line (so that the rope initially has length 30 feet). The rope has a density of 0.5 pounds per foot, and the supplies weigh 35 pounds.

a. [2 points] Suppose that Emily has lifted the supplies x feet from the ground (so that Emily has reeled in x feet of the rope). Find an expression in terms of x for the total weight, in pounds, of the supplies and the remaining length of rope being used to support the supplies.

Solution: The supplies weigh 35 pounds, and each foot of rope weighs 0.5 pounds. After the supplies have been lifted x feet, the total length of rope is $30 - x$ feet, and so the total weight, in pounds, is $35 + 0.5(30 - x)$.

Answer: $35 + 0.5(30 - x) = 50 - 0.5x$.

b. [4 points] Find an expression involving one or more integrals for the total work done in lifting the supplies 20 feet above the ground using the rope. Do **not** evaluate any integrals in your expression. Include units.

Solution: Using our expression from part a, we see that if the supplies have been lifted x feet already, then the work, in foot-pounds, done to raise the supplies a short extra distance Δx feet would be

$$(50 - 0.5x)\Delta x.$$

so the total work done in lifting the supplies from 0 to 20 feet above the ground is

$$\int_0^{20} (50 - 0.5x)\, dx.$$

Answer: $\int_0^{20} (50 - 0.5x)\, dx$

Answer: Units: **ft·lbs**

c. [4 points] After lifting the supplies 20 feet, the water tank in Emily's supplies bag is pierced, and begins to leak, so that the weight of the supplies decreases by 0.2 pounds per foot. Find an expression involving one or more integrals for the total work done in lifting the supplies the remaining distance of 10 feet using the rope. Do **not** evaluate any integrals in your expression. Include units.

Solution: We will present two different solution methods:
First, using our expression from part a. combined with the fact that the weight of the supplies decreases by 0.2 pounds for every foot it is raised above 20 feet, we see that the combined weight of the rope and supplies, for $20 \le x \le 30$ is

$$50 - 0.5x - 0.2(x - 20) = 54 - 0.7x.$$

Therefore, the work done to move a small extra distance Δx feet is

$$(54 - 0.7x)\Delta x,$$

and so the total work done is

$$\int_{20}^{30} (54 - 0.7x)\, dx.$$

Alternatively, the weight of the rope plus bag at height $20 + r$ is

$$\begin{aligned} \text{weight of supplies} + \text{weight of rope} &= (35 - 0.2r) + (30 - (20 + r))(0.5) \\ &= 40 - 0.7r \end{aligned}$$

So the work required to move an extra Δr feet would be

$$(40 - 0.7r)\Delta r$$

and r ranges from 0 to 10 feet, so the total work done is

$$\int_0^{10} 40 - 0.7r\, dr$$

Answer: $\int_{20}^{30} (54 - 0.7x)\, dx = \int_0^{10} 40 - 0.7r\, dr$

Answer: Units: **ft·lbs**

8. [8 points] Some of the values of a continuous, concave down, increasing function $g(x)$ are given in the following table:

x	0	1	2	3	4	5	6
$g(x)$	-17	-4	4	9	12	14	15

a. [3 points] Find the MID(3) approximation to

$$\int_0^6 g(x)\,dx.$$

Write out all the terms in your sum. You do not need to simplify, but your final answer should not involve the letter g.

Solution: The interval $[0,6]$ has width 6, so we should divide it into three subintervals of width 2. This means we should plug $x=1$, $x=3$, and $x=5$ into the function $g(x)$. We obtain:

$$\begin{aligned}\text{MID}(2) &= 2\left(g(1)+g(3)+g(5)\right)\\ &= 2(-4+9+14)\\ &= 2(19)\\ &= 38.\end{aligned}$$

Answer: 38

b. [2 points] Is the MID(3) estimate to $\int_0^6 g(x)\,dx$ you found in part (a) an underestimate, an overestimate, or is there not enough information (NEI)? Circle your choice. No justification is required.

Circle one: UNDERESTIMATE **OVERESTIMATE** NEI

c. [3 points] Find the RIGHT(3) approximation to

$$\int_0^6 3xg\left(\frac{x}{2}\right)\,dx.$$

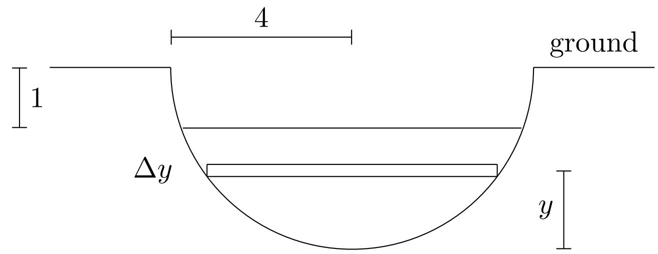
Write out all the terms in your sum. You do not need to simplify, but your final answer should not involve the letter g.

Solution: The interval $[0,6]$ has width 3, so we should divide it into three subintervals of width 2. This means we should plug $x=2$, $x=4$, and $x=6$ into the function $3xg\left(\frac{x}{2}\right)$. We obtain:

$$\begin{aligned}\text{RIGHT}(3) &= 2\left(3(2)g(1)+3(4)g(2)+3(6)g(3)\right)\\ &= 2(6(-4)+12(4)+18(9))\\ &= 2(-24+48+162)\\ &= 2(186)=372.\end{aligned}$$

Answer: 372

9. [8 points] Marcy's gigantic bird bath attracts lots of birds to her garden. The bath is carved out of the ground, in the shape of a **hemisphere** with radius 4 meters. A cross-section of the bath is depicted below. The bath is partially filled with muddy water, so that the surface of the water is 1 meter below ground level. The density of the water in the bath is given by the function $\delta(y)$ (measured in kilograms per cubic meter), where y is measured in meters from the **bottom of the bath**. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [4 points] Consider a horizontal slice of muddy water, y meters from the bottom of the bath with a small thickness of Δy meters, as depicted in the diagram above. Write an expression which approximates the mass, in kilograms, of this slice as a function of y. Your answer may include $\delta(y)$. Your answer should **not** involve any integrals.

Solution: Let r be the radius of the slice at height y. Using the Pythagorean Theorem, $r = \sqrt{16-(4-y)^2}$. Therefore, the mass of the slice is

$$\delta(y)\pi r^2 \Delta y = \delta(y)\pi\left(16-(4-y)^2\right)\Delta y = \delta(y)\pi\left(8y-y^2\right)\Delta y$$

Answer: $\delta(y)\pi\left(16-(4-y)^2\right)\Delta y = \delta(y)\pi\left(8y-y^2\right)\Delta y$

b. [4 points] Write an expression involving one or more integrals that gives the work done, in joules, to pump all the water in the bath up to ground level. Do not evaluate your integral(s).

Solution: Using our expression for the mass of a slice from part a., we see that the weight of a slice, in newtons, is:

$$\delta(y)\pi g\left(8y-y^2\right)\Delta y$$

A horizontal slice at height y will need to be lifted a distance of $4-y$ meters. Hence the work done to move such a slice is

$$\left(\delta(y)\pi g\left(8y-y^2\right)g\Delta y\right)(4-y) = \delta(y)\pi g\left(8y-y^2\right)(4-y)\Delta y$$

The water fills the bath up to height 3 meters, so the total work done to pump all the water in the bath up to ground level is

$$\int_0^3 \delta(y)\pi g\left(8y-y^2\right)(4-y)\,dy$$

Answer: $\int_0^3 \delta(y)\pi g\left(8y-y^2\right)(4-y)\,dy$

10. [10 points] The following parts are unrelated.

a. [3 points] Consider the region in the first quadrant bounded by $y = -(x-2)(x-7)$ and the x-axis. Which of the following expressions represents the **perimeter** of the region? Circle **one** option below.

i. $\int_2^7 -(x-2)(x-7)\,dx$

ii. $5 + \int \sqrt{1 + (-(x-2)(x-7))^2}\,dx$

iii. $5 + \int_2^7 \sqrt{1+(9-2x)^2}\,dx$

iv. $\int_2^7 \sqrt{1+(9-2x)^2}\,dx$

v. $\frac{1}{5}\int_2^7 -(x-2)(x-7)\,dx$

vi. NONE OF THESE

b. [4 points] Suppose that $f(x)$ is a differentiable function with a second derivative which is always positive. Suppose that $\text{LEFT}(40), \text{RIGHT}(40), \text{TRAP}(40)$, and $\text{MID}(40)$ are estimates of the integral $\int_0^{10} f(x)\,dx$. Which of the following **could** be true? Circle **all** options which apply.

i. $\text{LEFT}(40) < \int_0^{10} f(x)\,dx < \text{TRAP}(40)$

ii. $\text{TRAP}(40) < \int_0^{10} f(x)\,dx < \text{RIGHT}(40)$

iii. $\int_0^{10} f(x)\,dx < \text{TRAP}(40) < \text{LEFT}(40) < \text{RIGHT}(40)$

iv. $\text{MID}(40) < \int_0^{10} f(x)\,dx < \text{LEFT}(40) = \text{RIGHT}(40)$

v. $\text{TRAP}(40) = 180$ and the average value of $f(x)$ on the interval $[0, 10]$ is 20.

vi. $\text{TRAP}(40) = 220$ and the average value of $f(x)$ on the interval $[0, 10]$ is 20.

vii. NONE OF THESE

c. [3 points] Birds gather in a large area centered around Marcy's bird bath. The population density of birds, measured in birds per square kilometer, at a radial distance r kilometers from the center of Marcy's bird bath is given by the function $p(r)$.
Which of the following expressions must represent the total number of birds found within 5 kilometers of the center of the bird bath? Circle **one** option below.

i. $25\pi p(r)$

ii. $25\pi \int_0^5 p(r)\,dr$

iii. $\int_0^5 p(r)\,dr$

iv. $\int_{-5}^5 p(r)\,dr$

v. $\int_0^5 2\pi r p(r)\,dr$

vi. $\int_{-5}^5 2\pi r p(r)\,dr$

vii. $\int_0^5 \pi(p(r))^2\,dr$

viii. NONE OF THESE