Math 116 — Second Midterm — November 12, 2024

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 11 pages including this cover.
3. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	5	
3	7	
4	7	
5	9	
6	9	

Problem	Points	Score
7	7	
8	10	
9	12	
10	12	
11	12	
Total	100	

1. [10 points] Compute the **exact value** of each of the following. If a value **diverges, or otherwise does not exist**, write DNE. If there is **not enough information** to determine a given value, write NEI. You do not need to justify or simplify your answers.

a. [2 points] Find the value of p so that $\int_0^{10} \frac{1}{x^{2p}}\,dx$ and $\int_3^{\infty} \frac{1}{x^{2p}}\,dx$ both diverge.

Solution: For the first integral to diverge, we must have $2p \geq 1$. For the second integral to diverge we must have $2p \leq 1$. Therefore $2p = 1$, so $p = \frac{1}{2}$.

Answer: $p =$ 0.5

b. [2 points] Recall that a normal distribution has a probability density function (pdf) of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},$$

where μ is the mean of the distribution and σ is the standard deviation, with $\sigma > 0$. Find the exact value of

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-5)^2/18}\,dx.$$

Solution: Notice that for a normal distributions with mean 5 and standard deviation 3, the pdf is

$$\frac{1}{3\sqrt{2\pi}} e^{-(x-5)^2/18},$$

and since the area under a pdf is 1, we must have

$$\int_{-\infty}^{\infty} \frac{1}{3\sqrt{2\pi}} e^{-(x-5)^2/18}\,dx = 1.$$

Answer: 3

c. [2 points] Evaluate $\int_{-17}^{17} \frac{1}{x^2}\,dx$.

Solution: This integral diverges (by the p-test with $p = 2$).

Answer: **DNE**

d. [2 points] Find the exact value of the infinite sum $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots$.

Solution: This is an infinite geometric series with initial term 5 and common ratio $\frac{2}{3}$, so the infinite sum is $\frac{5}{1-\frac{2}{3}} = 15$.

Answer: 15

e. [2 points] Let $q(x)$ be a probability density function (pdf) for a statistic with mean value 5. Find the exact value of $\int_{-\infty}^{\infty} (1+x)q(x)\,dx$.

Solution: We have $\int_{-\infty}^{\infty} q(x)\,dx = 1$, and $\int_{-\infty}^{\infty} xq(x)\,dx = 5$, so $\int_{-\infty}^{\infty} (1+x)q(x)\,dx = \int_{-\infty}^{\infty} q(x)\,dx + \int_{-\infty}^{\infty} xq(x)\,dx = 6$.

Answer: 6

2. [5 points] Compute the following limit. Fully justify your answer including using **proper limit notation**.

$$\lim_{x\to\infty} 7x\ln\left(1+\frac{6}{x}\right)$$

Solution: We re-write the limit so that we may apply L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to\infty} 7x\ln\left(1+\frac{6}{x}\right) &= \lim_{x\to\infty}\frac{7\ln\left(1+\frac{6}{x}\right)}{1/x}\\
&\overset{\text{LH}}{=} 7\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{6}{x}}\right)\left(-\frac{6}{x^2}\right)}{-1/x^2}\\
&= 7\lim_{x\to\infty}\frac{6x^2}{(x^2)\left(1+\frac{6}{x}\right)}\\
&= 7\lim_{x\to\infty}\frac{6x^2}{x^2+6x}\\
&= 7(6)\\
&= 42
\end{aligned}$$

Answer: $\lim_{x\to\infty} 7x\ln\left(1+\frac{6}{x}\right) =$ 42

3. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_2^{10}\frac{1}{(t-2)^{1/3}}\,\mathrm{d}t$$

Solution:

$$\begin{aligned}
\int_2^{10}\frac{1}{(t-2)^{1/3}}\,\mathrm{d}t &= \lim_{a\to 2^+}\int_a^{10}\frac{1}{(t-2)^{1/3}}\,\mathrm{d}t\\
&= \lim_{a\to 2^+}\left(\frac{3}{2}(t-2)^{2/3}\right)\Big|_a^{10}\\
&= \frac{3}{2}(8)^{2/3} - \lim_{a\to 2^+}\frac{3}{2}(a-2)^{2/3}\\
&= 6
\end{aligned}$$

Circle one: **Diverges** [✓ **Converges to** 6]

4. [7 points] Determine whether the following improper integral converges or diverges and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use. You do not need to calculate the value of the integral if it converges.

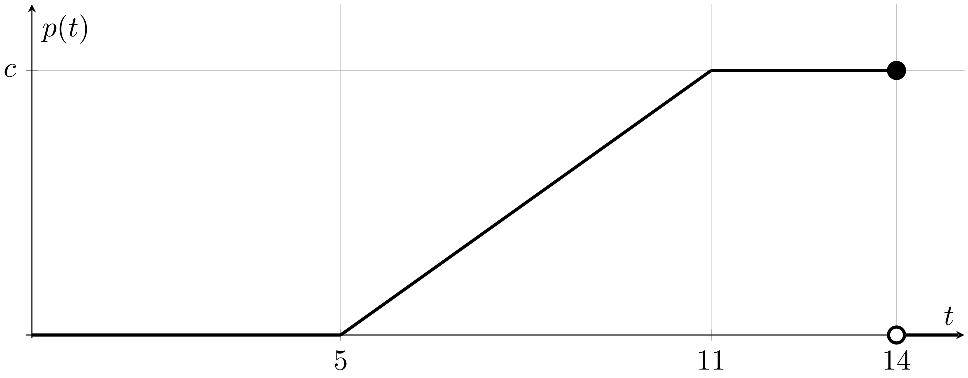
$$\int_1^\infty \frac{12+5\sin(x)}{x^{1/4}+x^{5/4}}\,\mathrm{d}x$$

Circle one: [✓ **Converges**] **Diverges**

Justification:

Solution: On the interval $1 \le x \le \infty$, we have $\dfrac{12+5\sin(x)}{x^{1/4}+x^{5/4}} \le \dfrac{17}{x^{5/4}}$, and $\displaystyle\int_1^\infty \frac{17}{x^{5/4}}\,\mathrm{d}x$ converges by the p-test with $p = \dfrac{5}{4}$. Therefore, by the (Direct) Comparison Test, $\displaystyle\int_1^\infty \frac{12+5\sin(x)}{x^{1/4}+x^{5/4}}\,\mathrm{d}x$ converges.

5. [9 points] Littorina the snail competes in a weekly race. The probability density function (pdf), $p(t)$ which describes the time in minutes it takes Littorina to finish the weekly race is depicted below. Note that $p(t)$ is piecewise linear, and that $p(t) = 0$ for $t < 5$ and $t > 14$.



a. [3 points] Find the value of c which makes $p(t)$ a probability density function.

Solution: For $p(t)$ to be a pdf, the area underneath the graph must be equal to 1. Therefore, $3c + 3c = 1$, and so $c = \frac{1}{6}$.

Answer: $c =$ $\frac{1}{6}$

b. [6 points] Find the function $P(t)$ which describes the probability that Littorina completes the weekly race in t minutes or less. Your formula should not contain any integral signs, but may include the letter c.

Solution:
The function $P(t)$ must be a continuous antiderivative of $p(t)$ which satisfies $\lim_{t\to-\infty} P(t) = 0$ and $\lim_{t\to\infty} P(t) = 1$. The slope of $p(t)$ on $5 \le t \le 14$ is $\frac{c}{6}$, and so we have:

$$P(t) = \begin{cases} 0, & t < 5 \\ \frac{c}{12}(t-5)^2, & 5 \le t < 11, \\ \frac{1}{2} + c(t-11), & 11 \le t < 14, \\ 1, & t \ge 14 \end{cases}$$

Plugging in $c = \frac{1}{6}$,

Answer: $$P(t) = \begin{cases} 0, & t < 5 \\ \frac{(t-5)^2}{72}, & 5 \le t < 11, \\ \frac{1}{2} + \frac{1}{6}(t-11), & 11 \le t < 14, \\ 1, & t \ge 14 \end{cases}$$

6. [9 points] As part of their training, once each week Littorina the snail attempts to travel as far as they can within one hour. Let $q(x)$ be the probability density function (pdf) that describes the total distance x, in centimeters, that Littorina manages to travel within an hour.

a. [3 points] Which of the following expressions represent the statement "*The median distance that Littorina travels in their training runs is 300 centimeters.*"? Circle **all** options which apply.

i. $q(300) = 0.5$

ii. $\int_0^\infty q(x)\ dx = 0.5$

iii. $\boxed{\int_0^{300} q(x)\ dx = 0.5}$

iv. $\int_0^\infty q(x)\ dx = 300$

v. $\int_0^\infty xq(x)\ dx = 300$

vi. $\int_0^\infty xq(x)\ dx = 0.5$

vii. $\boxed{\int_0^{300} q(x)\ dx = \int_{300}^\infty q(x)\ dx}$

viii. NONE OF THESE

b. [3 points] Circle the **one** statement below that is best supported by the equation

$$q(150) = 0.002.$$

i. The probability that Littorina travels exactly 150 centimeters is 0.002.

ii. Littorina travels 150 centimeters or fewer in approximately 0.2% of their training runs.

iii. Littorina travels between 150 and 151 centimeters in about 0.002% of their training runs.

iv. Every 150 seconds, Littorina travels roughly an extra 0.2 centimeters.

v. [✓ In their training runs, Littorina travels between 140 and 160 centimeters about 4% of the time.]

vi. NONE OF THE ABOVE.

c. [3 points] Let $Q(x)$ be the cumulative distribution function (cdf) which corresponds to $q(x)$. Suppose that 10% of the time, Littorina travels less than 90 centimeters. Additionally, suppose that 27% of the time, Littorina travels more than 500 centimeters. What is the value of $Q(500)-Q(90)$? Circle **one** option below.

i. 17

ii. 37

iii. 63

iv. 83

v. 0.17

vi. 0.37

vii. $\boxed{0.63}$

viii. 0.83

7. [7 points] Determine if the following series converges or diverges using the **Limit Comparison Test**, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2 - 3n + 4}}{n^2 - n + 1}$$

Circle one: **Converges** [✓ **Diverges**]

*Justification (using the **Limit Comparison Test**):*

Solution: We have

$$\lim_{n\to\infty} \left(\frac{\sqrt{2n^2 - 3n + 4}}{n^2 - n + 1} \right) \Big/ \left(\frac{1}{n} \right) = \lim_{n\to\infty} \frac{n\sqrt{2n^2 - 3n + 4}}{n^2 - n + 1} = \sqrt{2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1$), so $\sum_{n=1}^{\infty} \frac{\sqrt{2n^2 - 3n + 4}}{n^2 - n + 1}$ diverges by LCT.

8. [10 points] Determine if the following series converges absolutely, converges conditionally, or diverges, and circle the corresponding option. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^{0.3}}$$

Circle one: **Converges Absolutely** [✓ **Converges Conditionally**] **Diverges**

Justification:

Solution: Note that the series is alternating. Let $a_n = \frac{1}{n\,(\ln(n))^{0.3}}$. Then for all n, $0 < a_{n+1} < a_n$, and we also have $\lim_{n\to\infty} a_n = 0$. Therefore, by the alternating series test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln(n))^{0.3}}$ converges.

Now consider the series $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n\,(\ln(n))^{0.3}} \right| = \sum_{n=2}^{\infty} \frac{1}{n\,(\ln(n))^{0.3}}$.

Let $f(x) = \frac{1}{x\,(\ln(x))^{0.3}}$. Then for $x \geq 2$, $f(x)$ is positive and decreasing. We have:

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x\,(\ln(x))^{0.3}}\,\mathrm{d}x &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x\,(\ln(x))^{0.3}}\,\mathrm{d}x \\
&= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u^{0.3}}\,\mathrm{d}u \\
&= \lim_{b\to\infty} \frac{1}{0.7} u^{0.7} \Big|_{\ln(2)}^{\ln(b)} \\
&= \lim_{b\to\infty} \frac{1}{0.7} \left((\ln(b))^{0.7} - (\ln(2))^{0.7} \right) = \infty.
\end{aligned}$$

Therefore, $\int_2^{\infty} \frac{1}{x\,(\ln(x))^{0.3}}\,\mathrm{d}x$ diverges, and so by the Integral test, $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln(n))^{0.3}}$ diverges too.

Therefore $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln(n))^{0.3}}$ converges conditionally.

9. [12 points] Katydyd is on vacation from her strenuous bakery job, and is at the beach. She is building a tower out of sand, but periodically sand falls off the top of the tower. Each time sand falls off the tower it gets 25% shorter, and between times sand falls off the top of the tower Katydyd increases its height by 2 inches.

a. [5 points] Let M_n denote the height of Katydyd's tower, in inches, immediately *before* the n^{th} time sand falls off the top of it. Before the first time sand falls off the tower it has a height of 6 inches (so $M_1 = 6$). Find expressions for the values of M_2, M_3 and M_4. You do not need to simplify your expressions.

Solution:

$$\begin{aligned} M_1 &= 6 \\ M_2 &= 0.75M_1 + 2 = 0.75(6) + 2 \\ M_3 &= 0.75M_2 + 2 = 0.75^2(6) + 0.75(2) + 2 \\ M_4 &= 0.75M_4 + 3 = 0.75^3(6) + 0.75^2(2) + 0.75(2) + 2 \end{aligned}$$

Answer: $M_2 =$ $0.75(6) + 2$

Answer: $M_3 =$ $0.75^2(6) + 0.75(2) + 2$

Answer: $M_4 =$ $0.75^3(6) + 0.75^2(2) + 0.75(2) + 2$

b. [5 points] Find a closed-form expression for M_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression.

Solution:

$$\begin{aligned} M_n &= 0.75^{n-1}(6) + 2(0.75^{n-2} + \cdots + 0.75 + 1) \\ &= 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75} \end{aligned}$$

Answer: $M_n =$ $0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75}$

c. [2 points] If Katydyd were to keep doing this indefinitely, what height would her tower approach, in inches, in the long run?

Solution:

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} 0.75^{n-1}(6) + \frac{2(1 - 0.75^{n-1})}{1 - 0.75} = \frac{2}{0.25} = 8$$

Answer: **8**

10. [12 points]

a. [6 points] For each of the following sequences or series below, determine whether they must converge, must diverge, or whether there is not enough information. Circle your answers. No justification is required.

(i) $a_n = \int_1^n f(x)\,dx$ where $f(x) \geq 0$, $f'(x) \leq 0$, and $\lim\limits_{x\to\infty} f(x) = 0$.

Circle one: **Converges** **Diverges** [✓ **Not Enough Information**]

(ii) $\sum\limits_{n=1}^{\infty}(-1)^n(1+s^{-n})$ where s is a positive real number.

Circle one: **Converges** [✓ **Diverges**] **Not Enough Information**

(iii) $\sum\limits_{n=1}^{\infty}\frac{\sin n}{k^n}$ where k is a real number with $k > e$.

Circle one: [✓ **Converges**] **Diverges** **Not Enough Information**

b. [6 points] For each of the following sequences, defined for $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or not monotone, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $b_n = \frac{(-1)^n}{2n}$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** [✓ **Not Monotone**]

[✓ **Bounded**] **Unbounded**

(ii) $c_n = e^n \cos\left(\frac{1}{n}\right)$

*Circle **all** which apply:*

[✓ **Monotone Increasing**] **Monotone Decreasing** **Not Monotone**

Bounded [✓ **Unbounded**]

(iii) $d_n = \int_2^{2n} \frac{1}{(x-1)^2}\,dx$

*Circle **all** which apply:*

[✓ **Monotone Increasing**] **Monotone Decreasing** **Not Monotone**

[✓ **Bounded**] **Unbounded**

11. [12 points]

a. [7 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{10^n (n!)(n+1)!} x^{2n}$$

Solution: We use the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(2(n+1))!|x^{2(n+1)}|}{10^{n+1}(n+1)!(n+2)!} \cdot \frac{10^n (n!)(n+1)!}{(2n)!|x^{2n}|} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{10(n+1)(n+2)}|x^2| \\
&= \lim_{n\to\infty} \frac{4n^2}{10n^2}|x^2| \\
&= \frac{4}{10}|x^2|
\end{aligned}$$

The ratio test tells us the power series converges when this value is smaller than 1, i.e. $\frac{4}{10}|x^2| < 1$. Rearranging, we see that this implies $|x|^2 < \left(\frac{10}{4}\right)$, which tells us that the radius of convergence is $\sqrt{\frac{10}{4}}$.

Answer: $\frac{\sqrt{10}}{2}$

b. [5 points] No justification is needed for the remainder of this problem. Suppose that the following is true about the sequence C_n which is defined for $n \geq 0$:

- C_n is a monotone decreasing sequence of positive numbers which converges to 0.
- $\lim_{n\to\infty} \frac{C_n}{1/n} = 28.$
- The power series $\sum_{n=0}^{\infty} \frac{C_n}{4^n}(x-16)^n$ has radius of convergence 4.

What is the center of the interval of convergence of $\sum_{n=0}^{\infty} \frac{C_n}{4^n}(x-16)^n$?

Answer: 16

What are the endpoints of the interval of convergence of $\sum_{n=0}^{\infty} \frac{C_n}{4^n}(x-16)^n$?

Answer: Left endpoint at $c =$ 12

Right endpoint at $d =$ 20

Let c and d be the left and right endpoints of the interval of convergence you found above. Which of the following could be the interval of convergence of $\sum_{n=0}^{\infty} \frac{C_n}{4^n}(x-16)^n$? Circle **all** correct answers.

(c,d) $(c,d]$ [✓ $[c,d)$] $[c,d]$