Math 116 — First Midterm — September 30, 2025

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 11 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	14	
3	9	
4	6	
5	12	

Problem	Points	Score
6	9	
7	9	
8	8	
9	12	
10	10	
Total	100	

1. [11 points] Let $f(x)$ be a **twice-differentiable** function defined for all real numbers. Suppose that f also satisfies the following:

$$f(1) = -2, \qquad f(3) = 4, \qquad f(11) = 7,$$
$$\int_1^3 f(x)\,dx = 5, \qquad \int_3^{11} f(x)\,dx = 14$$

Compute the exact value of the following quantities. If there is not enough information provided to answer the question, write "NEI" and clearly indicate why. Show all of your work.

a. [3 points] The average value of $f(x)$ on the interval $[1, 11]$

Solution: The average value is

$$\frac{1}{11-1}\int_1^{11} f(x)\,dx = \frac{1}{10}\left(\int_1^3 f(x)\,dx + \int_3^{11} f(x)\,dx\right) = \frac{1}{10}(5+14) = 1.9.$$

Answer: 1.9

b. [4 points] $\int_1^3 y f'(y)\,dy$

Solution: Integrating by parts, we get

$$\int_1^3 y f'(y)\,dy = y f(y)\Big|_1^3 - \int_1^3 f(y)\,dy$$
$$= 12 + 2 - 5$$
$$= 9.$$

Answer: 9

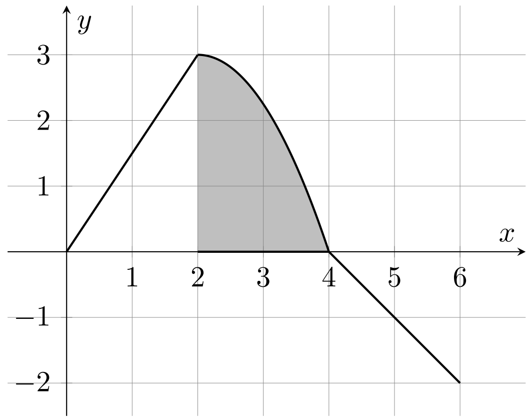
c. [4 points] $\int_1^3 x f(x^2+2)\,dx$

Solution: Using the substitution $u = x^2 + 2$, and remembering to change the bounds, we see:

$$\int_1^3 x f(x^2+2)\,dx = \frac{1}{2}\int_3^{11} f(u)\,du = 7.$$

Answer: 7

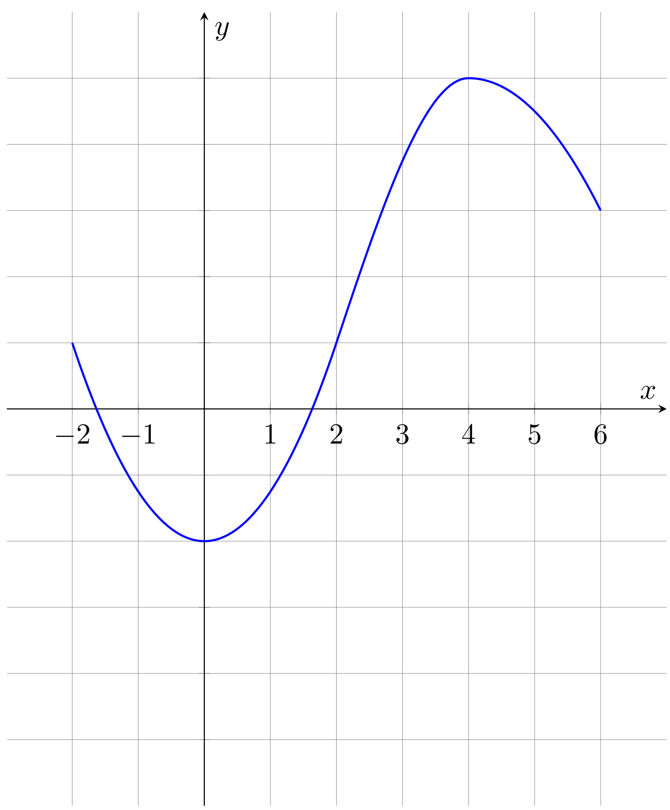
2. [14 points] A function $g(x)$, which is defined for all real numbers, is graphed on the interval $[0, 6]$ below. Note that $g(x)$ is linear on the intervals $(0, 2)$ and $(4, 6)$, and that the shaded region has area 4. Additionally, $g(x)$ is an **odd function**.



a. [4 points] The function $g(x)$ has a continuous antiderivative, $G(x)$, which satisfies $G(2) = 1$. Complete the following table of values for $G(x)$.

x	-2	0	2	4	6
$G(x)$	1	-2	1	5	3

b. [10 points] Sketch a graph of $G(x)$ on the interval $[-2, 6]$ using the axes provided. Make sure to clearly label the scale on the y-axis and also make it clear where $G(x)$ is increasing or decreasing, and where $G(x)$ is concave up, concave down, or linear.



3. [9 points] Consider the following function:

$$F(x) = 2 + \int_{-1}^{\sin(x)} \frac{1+t^2}{1+t^4}\,dt.$$

a. [2 points] Find a value of a such that $F(a) = 2$.

Solution: Since $\dfrac{1+t^2}{1+t^4} \geq 0$ for all t, we must have $F(x) \geq 2$. Note that if $\sin(a) = -1$, then we have

$$F(a) = 2 + \int_{-1}^{-1} \frac{1+t^2}{1+t^4}\,dt = 2 + 0 = 2.$$

So we want $\sin(a) = -1$. We choose $a = \dfrac{3\pi}{2}$, but

$$a = -\frac{\pi}{2} + 2k\pi$$

would work for any integer k.

Answer: $a =$ $3\pi/2$

b. [3 points] Calculate $F'(x)$.

Solution: We can write $F(x) = G(\sin(x))$ where

$$G(x) = 2 + \int_{-1}^{x} \frac{1+t^2}{1+t^4}\,dt$$

and we know from the Second Fundamental Theorem of Calculus that

$$G'(x) = \frac{1+x^2}{1+x^4}$$

Therefore, by the Chain Rule,

$$F'(x) = G'(\sin(x))(\sin(x))' = \cos(x)\frac{1+\sin^2(x)}{1+\sin^4(x)}.$$

Answer: $F'(x) =$ $\dfrac{\cos(x)(1+\sin^2(x))}{(1+\sin^4(x))}$

c. [4 points] Find a function $f(t)$ and constants a and C so that we may rewrite $F(x)$ in the form $\displaystyle\int_a^x f(t)\,dt + C$. There may be more than one correct answer.

Solution: Given the expression of $F'(x)$ we obtained in **b.** and that $F(3\pi/2) = 2$ as calculated in **a.**, we may write:

$$F(x) = 2 + \int_{3\pi/2}^{x} \frac{(1+\sin^2(t))\cos(t)}{1+\sin^4(t)}\,dt.$$

$f(t) =$ $\dfrac{(1+\sin^2(t))\cos(t)}{1+\sin^4(t)}$ $a =$ $\dfrac{3\pi}{2}$ $C =$ 2

4. [6 points] Use the partial fraction decomposition

$$\frac{9x^2 - 37x + 34}{(3-2x)(x-2)^2} = \frac{4}{(x-2)^2} - \frac{7}{x-2} - \frac{5}{3-2x}$$

to evaluate the following indefinite integral, showing all of your work. You do not need to simplify your answer.

$$\int \frac{9x^2 - 37x + 34}{(3-2x)(x-2)^2}\, dx$$

Solution: We first use the partial fraction decomposition, and then integrate (e.g. by using substitution in each part).

$$\int \frac{9x^2 - 37x + 34}{(3-2x)(x-2)^2}\, dx = \int \frac{4}{(x-2)^2}\, dx - \int \frac{7}{x-2}\, dx - \int \frac{5}{3-2x}\, dx$$
$$= -\frac{4}{x-2} - 7\ln|x-2| + \frac{5}{2}\ln|3-2x| + C$$

Answer: $-\frac{4}{x-2} - 7\ln|x-2| + \frac{5}{2}\ln|3-2x| + C$

5. [12 points] Melissa is an ecologist who finds that a new and increasingly popular brand of pesticide is harming native fish populations. She begins an information campaign about the dangers of the pesticide, and tracks $p(t)$, the **rate of change** of the concentration of pesticides in a local lake t days after the start of the campaign. The concentration of pesticides is measured in micrograms per liter ($\mu g/L$) so that the function $p(t)$ is measured in $\mu g/L$ per day.

Ten days after the start of the information campaign, Melissa finds that the concentration of pesticides in the lake is 360 micrograms per liter.

a. [4 points] Find an expression involving one or more integrals for the concentration, in micrograms per liter, of pesticides in the lake at the start of the information campaign. Your answer will involve the letter p.

Answer: $360 - \int_0^{10} p(t)\ dt$

Melissa finds the following table of values for $p(t)$ and $p'(t)$. She also discovers that $p'(t)$ is always decreasing for all $t > 0$.

t	2	4	6	8	10
$p(t)$	1.5	3	2	-1	-6
$p'(t)$	2	0	-1	-2	-3

b. [5 points] Melissa would like to find an **overestimate** to the integral

$$\int_2^{10} p(t)\, dt.$$

Which one of the following methods for approximating integrals would be guaranteed to give an overestimate?

Circle one: LEFT(n) RIGHT(n) MID(n) TRAP(n)

Find the approximation you chose above for the integral $\int_2^{10} p(t)\, dt$. You should use the maximal amount of equal subintervals possible, and write out all the terms in your sum.

Solution: The best we can do is MID(2), and the midpoints are 4, 8. Therefore, the approximation is $4p(4) + 4p(8) = 12 - 4 = 8$.

Answer: 8

c. [3 points] Recall the information about $p'(t)$ given above part **b.**, and that 10 days after the start of the information campaign, the pesticide concentration is 360 micrograms per liter. Melissa estimates the integral $\int_{10}^{20} p(t)\ dt$ using RIGHT(15) and finds that the result is -250. Which of the following statements **must** be true?

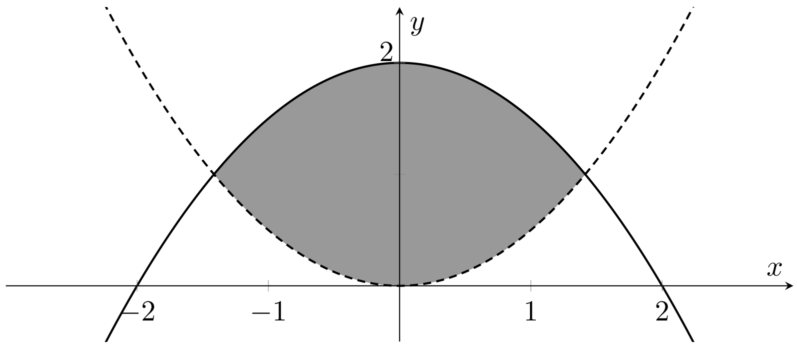
i. 20 days after the start of the information campaign, the concentration of pesticides must be **greater than** 110 micrograms per liter.

ii. 20 days after the start of the information campaign, the concentration of pesticides must be **less than** 110 micrograms per liter.

iii. 20 days after the start of the information campaign, the concentration of pesticides must be **equal to** 110 micrograms per liter.

iv. None of the above.

6. [9 points] A video game designer wants to model the shape of a mountain. The base of the mountain is the shaded region depicted below, bounded by the curves $y = 2 - \frac{x^2}{2}$ and $y = \frac{x^2}{2}$.



a. [5 points] Write an expression involving one or more integrals for the volume of a mountain whose base is the shaded region, and whose cross-sections perpendicular to the x-axis are semicircles. **Do not** evaluate any integrals in your expression.

Solution: First of all, x coordinates of points of intersection of the two curves are given by solutions of the equation $2 - \frac{x^2}{2} = \frac{x^2}{2}$ which are $x = -\sqrt{2}$, and $x = \sqrt{2}$.
Taking a slice x units from the y-axis, the *diameter* of the semicircle is given by

$$2 - \frac{x^2}{2} - \frac{x^2}{2} = 2 - x^2,$$

so the area of the semicircular prism with width Δx is

$$\frac{\pi}{2}\left(\frac{2 - x^2}{2}\right)^2 \Delta x.$$

Integrating from $-\sqrt{2}$ to $\sqrt{2}$ should then give the total volume.

Answer: $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{\pi}{2}\left(\frac{2 - x^2}{2}\right)^2 \, dx$

b. [4 points] Determine the perimeter of the shaded region. Write an expression that involves one or more integrals. **Do not** evaluate any integrals in your expression.

Solution: Exploiting symmetry, we calculate the arclength of one of the curves and multiply by 2. Let us denote the top curve by

$$f(x) = 2 - \frac{x^2}{2}$$

The arclength of the top curve is then

$$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + [f'(x)]^2} \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + x^2} \, dx,$$

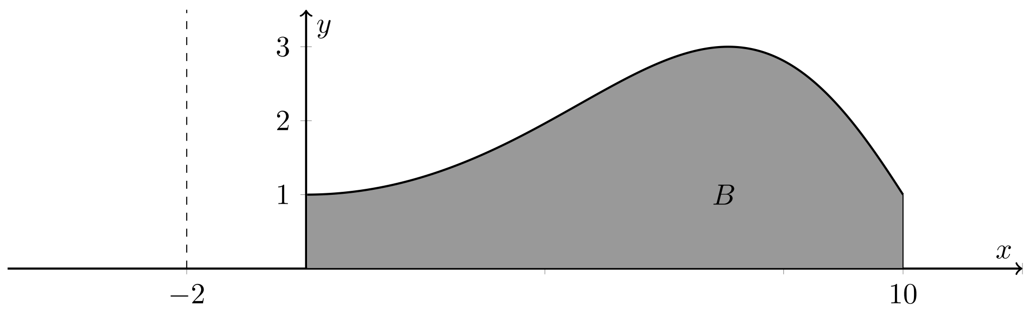
which we can double to find the total perimeter.

Answer: $2\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + x^2} \, dx$

7. [9 points] Consider the region B in the first quadrant, bounded by $x = 10$ and

$$f(x) = 1 + 2\sin\left(\frac{\pi x^2}{100}\right), \quad 0 \le x \le 10$$

as shaded in the diagram below.



a. [4 points] Pianta designs a beautiful decorative urn by rotating the shaded region B around the x-axis. Write an expression involving one or more integrals for the volume of the urn. Your expression should not involve the letter f. **Do not** evaluate any integrals in your expression.

Solution: We can slice the solid into disks with tiny horizontal width. For a disk located at x, the radius of the base is

$$f(x) = 1 + 2\sin\left(\frac{\pi x^2}{100}\right).$$

so the solid has volume

$$\int_0^{10} \pi\left[1 + 2\sin\left(\frac{\pi x^2}{100}\right)\right]^2 dx.$$

Answer: $\int_0^{10} \pi\left[1 + 2\sin\left(\frac{\pi x^2}{100}\right)\right]^2 dx$

b. [5 points] Pianta also designs a fountain basin for a city plaza by rotating the shaded region B around the line $x = -2$ depicted above. Write an expression involving one or more integrals for the volume of the fountain basin. Your expression should not involve the letter f. **Do not** evaluate any integrals in your expression.

Solution: We can slice the region B into horizontal slices, which after rotation become thin cylindrical shells. The volume of a thin shell formed by rotating a slice located at x is given by

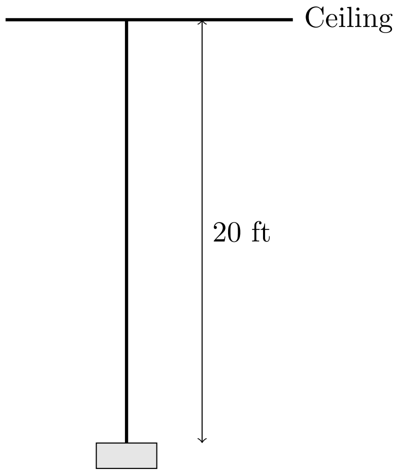
$$2\pi(2+x)f(x)\Delta x = 2\pi(2+x)\left[1 + 2\sin\left(\frac{\pi x^2}{100}\right)\right]\Delta x,$$

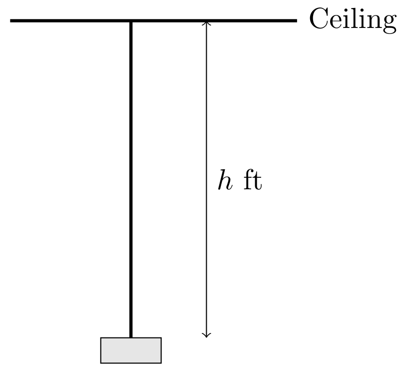
so the volume is given by

$$\int_0^{10} 2\pi(2+x)\left[1 + 2\sin\left(\frac{\pi x^2}{100}\right)\right] dx.$$

Answer: $\int_0^{10} 2\pi(2+x)\left[1 + 2\sin\left(\frac{\pi x^2}{100}\right)\right] dx$

8. [8 points] A theater technician is raising a pendant lamp. A chain of length 20 feet hangs from the ceiling and a lamp weighing 5 lbs is attached to the bottom of the chain, as depicted below to the left. The density of the chain is 0.6 lb/ft. The technician pulls the lamp and chain straight upward toward the ceiling. As the lamp is lifted, the technician does not need to lift the portion of the chain that has already been "reeled in", that is, the part that has reached the ceiling. The lamp is lifted in this manner until it is 12 feet above its initial position.





a. [4 points] Suppose the lamp has been lifted until it is h feet from the ceiling, as shown in the diagram on the top right. Write an expression that approximates the work required to lift the chain and the lamp by a small distance Δh feet. Your answer should not involve any integrals, and should be expressed in terms of h and Δh. Include units.

Solution: The weight of the rope plus the lamp is given by

$$0.6h + 5 \quad \text{ft} \cdot \text{lb},$$

so the work done is

$$(0.6h + 5)\Delta h \quad \text{ftlbs}.$$

Answer: $(0.6h + 5)\Delta h$ **Units:** ftlbs

b. [4 points] Write an expression involving one or more integrals that represents the total work required to raise the lamp 12 feet above its initial position using the chain. **Do not** evaluate any integrals in your expression. Include units.

Solution: The lamp is initially 20 ft from the ceiling, and when the lamp is at its final position, it is 8 ft below the ceiling. Therefore, the total work done is

$$\int_8^{20} (0.6h + 5)\ dh \quad \text{ftlbs}.$$

Answer: $\int_8^{20} (0.6h + 5)\ dh$ **Units:** ftlbs

9. [12 points]

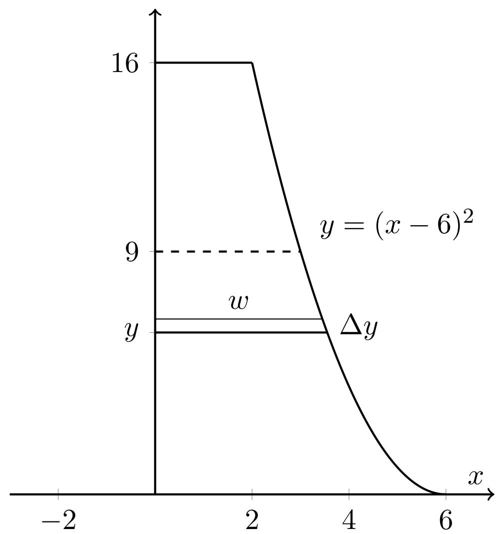
A big tank at a chemical factory is formed by rotating the region in the first quadrant bounded by $y = 16$, and

$$y = (x-6)^2,$$

around the y-axis. All distances are measured in meters. The tank is filled with liquid chemicals up to $y = 9$ meters, as shown by the dashed line in the plot to the right. Due to sedimentation, the liquid has a varying density of

$$f(y) = 3 - 0.1y \quad \text{kg/m}^3$$

at height y. Workers at the factory will pump the chemicals out through the top of the tank. You may assume that the acceleration due to gravity is $g = 9.8\text{m/s}^2$.



a. [2 points] Consider the thin horizontal strip of the region depicted above, which is located y meters above the x-axis. It has horizontal length w and a small thickness Δy. Find a formula for w in terms of y.

Solution: We know that $y = (w-6)^2$ and $w \le 6$, so $w = 6 - \sqrt{y}$.

Answer: $w =$ $6 - \sqrt{y}$

b. [4 points] When the strip above is rotated around the y-axis, it forms a thin **disk**. Write an expression which approximates the **mass** of that disk. Your answer should not involve any integrals, and you should express your answer in terms of y, and Δy. **Include units.**

Solution: The volume of a slice is $\pi w^2 \Delta y$. To find the mass of a slice, we must multiply by the density, giving us $\pi w^2 f(y)\Delta y = \pi(6-\sqrt{y})^2(3-0.1y)\Delta y$.

Answer: $\pi(6-\sqrt{y})^2(3-0.1y)\Delta y$ **Units:** kg

c. [3 points] Write an expression which approximates the work needed to lift the thin disk described in part **b** to the top of the tank. Your answer should not involve any integrals, and you should express your answer in terms of y, and Δy. **Include units.**

Solution: We multiply the mass of the slice by g to get its weight, and then multiply by the distance it travels to get the work done on the slice. Therefore, the work done is $\pi(6-\sqrt{y})^2(3-0.1y)(9.8)(16-y)\Delta y$.

Answer: $9.8\pi(6-\sqrt{y})^2(3-0.1y)(16-y)\Delta y$ **Units:** Joules

d. [3 points] Write an expression involving one or more integrals representing the work needed to pump all the liquid chemicals to top of the tank, using the same units as in part **c**. **Do not** evaluate any integrals in your expression.

Solution: We note that the lower bound of y should be 0, and the upper bound of y should be 9. Thus the total work done is $\int_0^9 9.8\pi(6-\sqrt{y})^2(3-0.1y)(16-y)\,dy$ Joules.

Answer: $\int_0^9 9.8\pi(6-\sqrt{y})^2(3-0.1y)(16-y)\,dy$

10. [10 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer. Justification is not required.

a. [2 points] If $a(x)$ is a concave up differentiable function, and RIGHT(8) and TRAP(8) are used to estimate $\int_{-1}^{1} a(x)\, dx$, then

$$\text{RIGHT}(8) < \text{TRAP}(8).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

b. [2 points] If $b(x)$ is an increasing, concave down differentiable function, and TRAP(10) and MID(10) are used to estimate $\int_{-1}^{1} b(x)\, dx$, then

$$\text{MID}(10) < \int_{-1}^{1} b(x)\, dx < \text{TRAP}(10).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

c. [2 points] Suppose that $f(x)$ is a decreasing differentiable function, and that LEFT(2) and LEFT(4) are used to estimate $\int_{-1}^{1} f(x)\, dx$. Then

$$\int_{-1}^{1} f(x)\, dx \le \text{LEFT}(4) \le \text{LEFT}(2).$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

d. [2 points] Suppose that $g(x)$ is a continuous function with an antiderivative $G(x)$ which satisfies $G(3) = 5$. Suppose that $\int_{3}^{7} g(t)\, dt = 4$, and let $H(x) = \int_{7}^{x} g(t)\, dt$. Then $G(50) - H(50) = 9$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

e. [2 points] Suppose that $h(x)$ is a continuous odd function. Then

$$\int_{-1}^{1} x^2 \left(h(x)\right)^2\, dx = 2\int_{0}^{1} x^2 \left(h(x)\right)^2\, dx.$$

Circle one: **ALWAYS** **SOMETIMES** **NEVER**