Math 116 — Second Midterm — November 11, 2025

EXAM SOLUTIONS

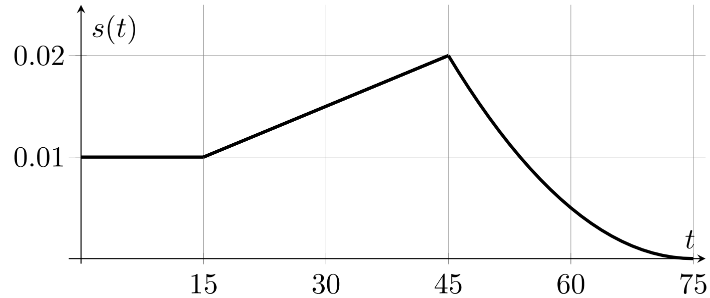
1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 12 pages including this cover.
3. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	9	
3	8	
4	7	
5	7	

Problem	Points	Score
6	12	
7	15	
8	12	
9	10	
10	11	
Total	100	

1. [9 points] Steph times their study sessions. Let t be the length in minutes, and let $s(t)$ be the probability density function for t. A **portion** of the graph of $s(t)$ is shown below. Also note that $s(t)$:

- is continuous;
- is constant for $0 \le t \le 15$;
- is linear for $15 \le t \le 45$;
- is concave up for $45 \le t \le 75$.



a. [3 points] Find the proportion of Steph's study sessions that last between 15 minutes and 45 minutes. You do not need to simplify your answer but you should show your work.

Solution: The proportion is equal to

$$\int_{15}^{45} s(t)\,dt = \frac{1}{2} \cdot 30 \cdot (0.01 + 0.02) = 0.45.$$

Answer: 0.45

b. [3 points] Which of the statements below is best supported by the expression $s(55) \approx 0.01$? Circle the **one** best statement. No justification is required.

i. Approximately 0.01% of sessions last *exactly* 55 minutes.

ii. Approximately 1% of sessions last between 54 and 56 minutes.

iii. $\boxed{\text{Approximately 2\% of sessions last between 54 and 56 minutes.}}$

iv. Approximately 10% of sessions last between 55 and 56 minutes.

v. Approximately 55% of sessions last around 0.01 minutes.

c. [3 points] Do any sessions last longer than 75 minutes? Give a brief explanation of your answer.

Circle one: $\boxed{\text{Yes}}$ No Not enough information

Explanation:

Solution: We can compute $\int_0^{45} s(t)\,dt = 0.6$, and since $s(t)$ is concave up for $45 \le t \le 75$, we have $\int_{45}^{75} s(t)\,dt \le 0.3$ (equality would hold if $s(t)$ was linear for $45 \le t \le 75$). Therefore, $\int_0^{75} s(t)\,dt \le 0.9$. Since s is a pdf, we know that $\int_0^{\infty} s(t)\,dt = 1$, so the above graph is not a complete graph of $s(t)$ and there must be sessions longer than 75 minutes.

2. [9 points] Let t (in minutes) denote the time Audrey waits for the *Bursley-Baits* shuttle to arrive. Observations show that the **probability density function** (pdf) of her wait time (in minutes) is of the form

$$p(t) = \begin{cases} 0, & t < 0, \\ 2\lambda t e^{-\lambda t^2} & t \geq 0, \end{cases}$$

where λ is a positive constant.

Throughout this problem, show all your work, and write your answers in exact form.

a. [5 points] Suppose that Audrey's median wait time for the Bursley-Baits shuttle is 1 minute. Find the value of λ.

Solution: If we call the cumulative distribution function (cdf) $P(x)$, then

$$\begin{aligned} P(x) &= \int_0^x p(t)\ dt \\ &= \int_0^x 2\lambda t e^{-\lambda t^2}\ dt \\ &= \int_0^{\lambda x^2} e^{-u}\ du \\ &= 1 - e^{-\lambda x^2}. \end{aligned}$$

Therefore, solving $P(1) = 0.5$ we get $\lambda = \ln 2$.

Answer: $\ln 2$

Sometimes Audrey takes the *Northwood Express* shuttle. For the Northwood Express shuttle, Audrey's wait time (in minutes) follows a **cumulative distribution function** (cdf) of the form

$$Q(t) = \begin{cases} 0, & t < 0, \\ 1 - (\lambda t + 1)e^{-\lambda t} & t \geq 0, \end{cases}$$

where λ is the **same** as in part **a**.

b. [2 points] When Audrey takes the Northwood Express shuttle, what is the fraction of rides where Audrey waits for 1 minute or less for the shuttle to arrive? Your final answer should not involve λ.

Solution: We have

$$Q(1) = 1 - (\lambda + 1)e^{-\lambda} = \frac{1}{2} - \frac{1}{2}\ln 2.$$

Answer: $\frac{1}{2} - \frac{1}{2}\ln 2$

c. [2 points] Audrey wants to choose the shuttle that has a lower median wait time. Which one should she choose? Explain your answer.

Circle one: $\boxed{\text{THE BURSLEY-BAITS SHUTTLE}}$ THE NORTHWOOD EXPRESS SHUTTLE

Explanation:

Solution: Since $Q(1) < 0.5$, we know that the median of the wait time for Northwood shuttle is greater than 1, while the median of the wait time for the Bursley-Baits shuttle is equal to 1. Therefore, Audrey should choose the Bursley-Baits shuttle.

3. [8 points]

a. [5 points] Show that the following limit converges and compute the limit. Fully justify your answer including using **proper limit notation**.

$$\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$$

Solution:
We use L'Hospital's Rule twice to find this limit.

$$\begin{aligned}
\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right) &= \lim_{n\to\infty} \frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \\
&\overset{\text{LH}}{=} \lim_{n\to\infty} \frac{\sin(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{2}{n^3}} \\
&= \frac{1}{2}\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \\
&= \frac{1}{2}\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \\
&\overset{\text{LH}}{=} \frac{1}{2}\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) \\
&= \frac{1}{2}
\end{aligned}$$

Answer: $\frac{1}{2}$

b. [3 points] Use part **a.** to determine if the following series converges or diverges, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$$

Circle one: **Converges** **Diverges**

Justification:

Solution: From **a.** we know that

$$\lim_{n\to\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right) = \frac{1}{2} \neq 0$$

and so by the nth term test for divergence, $\sum_{n=1}^{\infty} n^2\left(1-\cos\left(\frac{1}{n}\right)\right)$ diverges.

4. [7 points] **Compute** the value of the following improper integral if it converges.
If it does not converge, use a **direct computation** of the integral to show its divergence.
Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_0^{\sqrt{8}} \frac{x}{(8-x^2)^{1/3}}\,\mathrm{d}x$$

Solution:
We first rewrite this improper integral as a limit.

$$\int_0^{\sqrt{8}} \frac{x}{(8-x^2)^{1/3}}\,\mathrm{d}x = \lim_{b\to\sqrt{8}^-} \int_0^{b} \frac{x}{(8-x^2)^{1/3}}\,dx$$

We then use the substitution $u = 8 - x^2$ to see,

$$\begin{aligned}
\lim_{b\to\sqrt{8}^-} \int_0^{b} \frac{x}{(8-x^2)^{1/3}}\,dx &= \lim_{b\to\sqrt{8}^-} -\frac{1}{2}\int_8^{8-b^2} \frac{du}{u^{1/3}}\,dx \\
&= \lim_{b\to\sqrt{8}^-} -\frac{3}{4}u^{2/3}\Big|_8^{8-b^2} \\
&= \lim_{b\to\sqrt{8}^-} \frac{3}{4}\left(8^{2/3} - (8-b^2)^{2/3}\right) \\
&= \frac{3}{4}(4-0) = 3.
\end{aligned}$$

Therefore the integral converges to 3.

Circle one: **Diverges** $\boxed{\textbf{Converges to } \underline{\quad 3 \quad}}$

5. [7 points] Determine whether the following improper integral converges or diverges and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use. You do not need to calculate the value of the integral if it converges.

$$\int_2^\infty \frac{x+1}{5x^3+3x^3\sin x}\,\mathrm{d}x$$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: We use Direct Comparison Test (DCT) to show the convergence of the integral. Note that for $x \geq 2$,

$$\frac{x+1}{5x^3+3x^3\sin x} \leq \frac{2x}{2x^3} = \frac{1}{x^2}$$

and that by p-test (with $p = 2$),

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges. Therefore, by Direct Comparison Test,

$$\int_2^\infty \frac{x+1}{5x^3+3x^3\sin x}\,\mathrm{d}x$$

also converges.

6. [12 points]

a. [6 points] For each of the following sequences or series described below, defined for $n \geq 1$, determine whether they must converge, must diverge, or whether there is not enough information. Circle your answers. No justification is required.

(i) $a_n = (-1)^n(2 + k^{-n})$, where k is a positive real number.

Circle one: **Converges** $\boxed{\textbf{Diverges}}$ **Not Enough Information**

(ii) $b_n = \displaystyle\int_2^{n+3} f(x)\,dx$ where $f(x)$ is a positive function, and the series $\displaystyle\sum_{j=2}^{\infty} f(j)$ converges.

Circle one: **Converges** **Diverges** $\boxed{\textbf{Not Enough Information}}$

(iii) $c_n = P(e^n)$ where $P(x)$ is a cumulative distribution function.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges** **Not Enough Information**

b. [6 points] For each of the following sequences, defined for $n \geq 1$, decide whether the sequence is monotone increasing, monotone decreasing, or not monotone, and whether it is bounded or unbounded. Circle your answers. No justification is required.

(i) $r_n = \cos(2\pi n)\left(\dfrac{5}{4}\right)^n$

*Circle **all** which apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

Bounded $\boxed{\textbf{Unbounded}}$

(ii) $s_n = \dfrac{(-1)^n}{1 + \ln(n)}$

*Circle **all** which apply:*

Monotone Increasing **Monotone Decreasing** $\boxed{\textbf{Not Monotone}}$

$\boxed{\textbf{Bounded}}$ **Unbounded**

(iii) $t_n = \displaystyle\int_1^{n^3} 2^{-x}\,dx$

*Circle **all** which apply:*

$\boxed{\textbf{Monotone Increasing}}$ **Monotone Decreasing** **Not Monotone**

$\boxed{\textbf{Bounded}}$ **Unbounded**

7. [15 points]

a. [7 points] Determine if the following series converges or diverges using the **Limit Comparison Test**, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{3n-2}{\sqrt{4n^3-5n^2+3}}$$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Justification (using the ***Limit Comparison Test****):*

Solution: We aim to compare with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. We see that

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{3n-2}{\sqrt{4n^3-5n^2+3}}}{\frac{1}{n^{1/2}}} &= \lim_{n\to\infty} \frac{3n^{3/2}-2n^{1/2}}{\sqrt{4n^3-5n^2+3}} \\ &= \lim_{n\to\infty} \frac{3n^{3/2}}{2n^{3/2}} \\ &= \frac{3}{2},\end{aligned}$$

and so the Limit Comparison Test does apply.
By the p-test for series (with $p = \frac{1}{2}$),

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

diverges. Therefore, by the Limit Comparison Test,

$$\sum_{n=1}^{\infty} \frac{3n-2}{\sqrt{4n^3-5n^2+3}}$$

also diverges.

b. [8 points] Determine if the following series absolutely converges, conditionally converges, or diverges, and circle the corresponding word. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}+1}$$

Circle one: **Converges Absolutely** $\quad$ **Converges Conditionally** $\quad$ **Diverges**

Justification:

Solution: First of all, note that

$$\frac{1}{\sqrt{n}+1} \geq \frac{1}{2\sqrt{n}}$$

for $n \geq 1$. Since

$$\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$$

diverges by the p-test for series (with $p = \frac{1}{2}$), we see by the Comparison Test that

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$$

diverges.

On the other hand, we note that

$$\frac{1}{\sqrt{n}+1}$$

is monotone decreasing and approaches 0 as $n \to \infty$, and so by the Alternating Series Test (AST),

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}+1}$$

converges.

As a result, $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}+1}$ **converges conditionally**.

8. [12 points] Melissa runs an operation to remove pesticides from a local lake. Initially there are 200kg of pesticides in the lake. The operation runs from 10am till 10pm each day, and by 10pm each day, the mass of pesticides in the lake is always $p\%$ lower than it was at 10am, where p is a constant. From 10pm till 10am each night, runoff from local rivers adds 12kg of pesticides back into the lake.

a. [5 points] Let S_n denote the total mass of pesticides, in kg, in the lake at 10pm on the nth day of operation. Find expressions for the values of S_1, S_2 and S_3. You do not need to simplify your expressions, and they may be given in terms of p.

Solution: We see that $S_1 = 200\left(1-\frac{p}{100}\right) = 200 - 2p$. Therefore $S_2 = (S_1+12)\left(1-\frac{p}{100}\right) = 200\left(1-\frac{p}{100}\right)^2 + 12\left(1-\frac{p}{100}\right)$, and so $S_3 = (S_2 + 12)\left(1-\frac{p}{100}\right) = 200\left(1-\frac{p}{100}\right)^3 + 12\left(1-\frac{p}{100}\right)^2 + 12\left(1-\frac{p}{100}\right)$.

Answer: $S_1 =$ $200\left(1-\frac{p}{100}\right)$

Answer: $S_2 =$ $200\left(1-\frac{p}{100}\right)^2 + 12\left(1-\frac{p}{100}\right)$

Answer: $S_3 =$ $200\left(1-\frac{p}{100}\right)^3 + 12\left(1-\frac{p}{100}\right)^2 + 12\left(1-\frac{p}{100}\right)$

b. [5 points] Find a closed-form expression for S_n. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. You do not need to simplify your expression and it may be given in terms of p.

Solution: Following our intuition from part **a.**, we can see that

$$S_n = 200\left(1-\frac{p}{100}\right)^n + 12\left(1-\frac{p}{100}\right)^{n-1} + \cdots + 12\left(1-\frac{p}{100}\right).$$

Using the formula for the sum of a finite geometric series,

$$\begin{aligned} S_n &= 200\left(1-\frac{p}{100}\right)^n + \frac{12\left(1-\frac{p}{100}\right)\left(1-\left(1-\frac{p}{100}\right)^{n-1}\right)}{1-\left(1-\frac{p}{100}\right)} \\ &= 200\left(1-\frac{p}{100}\right)^n + \frac{1200}{p}\left(1-\frac{p}{100}\right)\left(1-\left(1-\frac{p}{100}\right)^{n-1}\right) \end{aligned}$$

Answer: $S_n =$ $200\left(1-\frac{p}{100}\right)^n + \frac{1200}{p}\left(1-\frac{p}{100}\right)\left(1-\left(1-\frac{p}{100}\right)^{n-1}\right)$

c. [2 points] Melissa aims to make sure that, in the long run, the mass of pesticides in the lake at 10pm each day approaches 28kg. What is the smallest value of p that will ensure that Melissa meets her goal? Show all of your work.

Solution: We note that $\lim\limits_{n\to\infty} S_n = \frac{1200}{p}\left(1-\frac{p}{100}\right) = \frac{1200}{p} - 12$. We want $\frac{1200}{p} - 12 = 28$, and thus $p = 30$.

Answer: $p =$ 30

9. [10 points]

a. [7 points] Determine the **radius** of convergence for the following power series. Show all of your work. You do not need to find the interval of convergence.

$$\sum_{n=1}^{\infty}(-1)^n \frac{(3n)!}{5^n((n+1)!)^3}(x-4)^{2n}$$

Solution: We denote

$$a_n = (-1)^n \frac{(3n)!}{5^n((n+1)!)^3}(x-4)^{2n}.$$

We see that:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(3n+3)!}{5^{n+1}((n+2)!)^3}\frac{5^n((n+1)!)^3}{(3n)!}\frac{|x-4|^{2n+2}}{|x-4|^{2n}} \\ &= \lim_{n\to\infty}\frac{5^n}{5^{n+1}}\frac{(3n+3)!}{(3n)!}\frac{((n+1)!)^3}{((n+2)!)^3}|x-4|^2 \\ &= \lim_{n\to\infty}\frac{1}{5}\frac{(3n+3)(3n+2)(3n+1)}{(n+2)^3}|x-4|^2 \\ &= \frac{27}{5}|x-4|^2,\end{aligned}$$

and so by the Ratio test, the power series converges for $\frac{27}{5}|x-4|^2 < 1$, i.e. for $|x-4| < \left(\frac{5}{27}\right)^{1/2}$.

Similarly, the power series diverges for $|x-4| > \left(\frac{5}{27}\right)^{1/2}$.

Therefore, the radius of convergence is $\left(\frac{5}{27}\right)^{1/2}$.

Answer: $\left(\frac{5}{27}\right)^{1/2}$

b. [3 points] Suppose the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ has radius of convergence 3, and that the series diverges for $x=7$ and converges for $x=10$. Which of the following could be the value of a? Circle **all** correct options.

i. 4 ii. 7 iii. $\boxed{10}$ iv. $\boxed{13}$ v. 16 vi. NONE OF THESE

10. [11 points] The following parts are unrelated. Throughout this problem, justification is not required.

a. [3 points] Which of the following represent power series in the variable x? Circle **all** options which apply.

i. $\boxed{x + x^3 + x^7 + x^{10} + \cdots}$

ii. $\dfrac{1}{x-1} + \dfrac{1}{(x-1)^2} + \dfrac{1}{(x-1)^3} + \cdots$

iii. $1 + (x-1)^2 + (x-2)^3 + (x-3)^4 + \cdots$

iv. $\boxed{x^{10} + 3x^3 + 12}$

v. $\displaystyle\sum_{n=0}^{\infty} (\sin x)^n$

vi. $\boxed{\displaystyle\sum_{n=0}^{\infty} 2^n x^n}$

vii. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n/2}}{n}$

viii. None of these

b. [2 points] Suppose $p(x)$ is a probability density function (pdf) for a statistic which has mean value 4. Find the exact value of $\displaystyle\int_{-\infty}^{\infty} (2x+3)p(x)\, \mathrm{d}x$.

Answer: 11

c. [6 points] For the following questions, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true, and circle the corresponding answer.

(i) Let $g(x)$ be a positive, decreasing, and continuous function. Suppose that for all n, $s_n = \displaystyle\int_n^{n+1} g(x)\, \mathrm{d}x$. Then the sequence s_n converges.

Circle one: **$\boxed{\text{ALWAYS}}$** **SOMETIMES** **NEVER**

(ii) Let $b_n \geq 0$ and $c_n \geq 0$ for all n. Suppose that the series $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and that the sequence c_n diverges. Then the series $\displaystyle\sum_{n=1}^{\infty} b_n c_n$ diverges.

Circle one: **ALWAYS** **$\boxed{\text{SOMETIMES}}$** **NEVER**

(iii) Let $F(x)$ be a cumulative distribution function (cdf) that is continuous for all x. Then the series $\displaystyle\sum_{n=1}^{\infty} (F(n+1) - F(n))$ converges.

Circle one: **$\boxed{\text{ALWAYS}}$** **SOMETIMES** **NEVER**