Math 116 - Exam I

Department of Mathematics
University of Michigan
February 6, 2003

Name: SOLUTIONS

Instructor: ____________________ Section Number: __________

1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are 10 questions. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full numeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a 3" by 5" notecard.
7. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	6	
3	12	
4	6	
5	10	
6	10	
7	10	
8	10	
9	14	
10	10	
TOTAL	100	

1. (3 pts each) Circle true or false. No explanation necessary.

   (a) If $0 < f(x) < g(x)$ for all x, then $\int_1^4 \frac{g(x)}{f(x)} > 3$.

   True $\int_1^4 \frac{g(x)}{f(x)}\,dx > \int_1^4 \frac{f(x)}{f(x)}\,dx = \int_1^4 1\,dx = x|_1^4 = 4 - 1 = 3.$

   (b) $\int_\pi^\pi \frac{\sin^2(x^5) - \cos(42x)}{x^2 + x + 1} > 2$.

   False The integral is from π to π, so it's 0.

   (c) $\int_1^\infty \frac{1}{x^{71}}\,dx$ converges.

   True $\int_1^\infty \frac{1}{x^{71}}\,dx = \lim_{b\to\infty} \int_1^b x^{-71}\,dx = \lim_{b\to\infty} -\frac{1}{70}x^{-70}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{70b^{70}} + \frac{1}{70} = \frac{1}{70}.$

   (d) If $g(x) + 2$ is an antiderivative of $f(x)$, then $g(x)$ is also an antiderivative of $f(x)$.

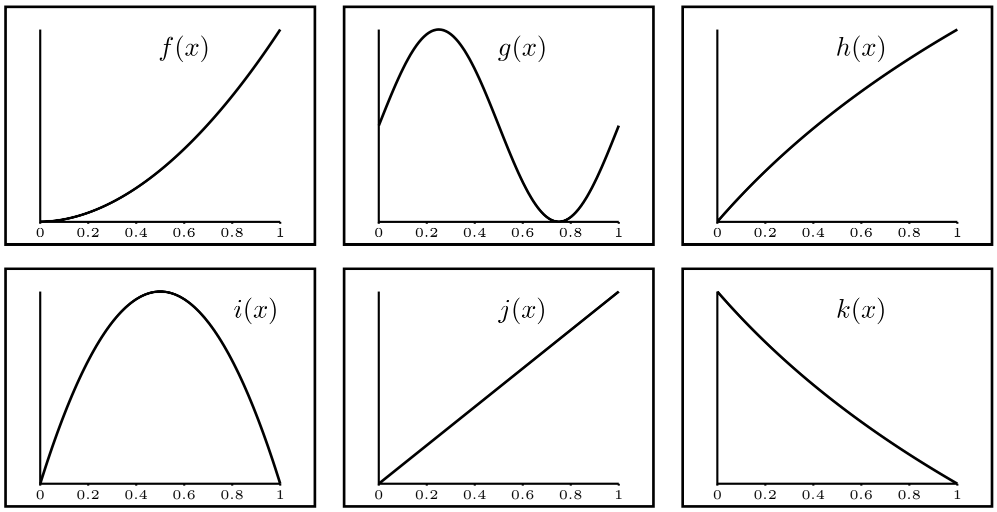
   True $g(x) + 2$ and $g(x)$ have the same derivative.

2. (6 pts) State the construction theorem for antiderivatives.

   If $f(x)$ is continuous on the closed interval $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ for $a \le x \le b$.

3. (12 pts) Below are the graphs of several functions $f(x)$, $g(x)$, $h(x)$, $i(x)$, $j(x)$, and $k(x)$. Do not assume that the y-axis scales on these graphs are equal or even comparable. We have calculated LEFT(6), RIGHT(6), TRAP(6), and MID(6) for four of these six functions. Label each column with the name of the function estimated in that column.

Function:	$j(x)$	$h(x)$	$f(x)$	$g(x)$
LEFT(6):	64.2	.328	.255	80.0
RIGHT(6):	65.8	.444	.421	80.0
TRAP(6):	65.0	.386	.338	80.0
MID(6):	65.0	.388	.331	80.0



Consider the rightmost column, with all the estimates the same. Since LEFT(6) = RIGHT(6), it can't be increasing or decreasing, which eliminates f, h, j, and k. Since TRAP(6) = MID(6), it can't be entirely concave up or concave down, which eliminates i. So it's g. In all of the other columns, LEFT(6) $<$ RIGHT(6), which eliminates k (deceasing, so LEFT(6) $>$ RIGHT(6)) and i ($i(0) = i(1)$, so LEFT(6) = RIGHT(6)). That leaves f, j, and h, and we can tell the difference between them by their concavity, which dictates the relationship between MID(6) and TRAP(6).

4. (6 pts) Write an integration problem (of your choice) for which the substitution $w = 1/x$ would be the best way to start. You need not evaluate your own integral.

   Many choices here. A good one will have $1/x$ inside some other function, and its derivative (up to a constant) outside. So
$$\int \frac{\sin(1/x)}{x^2}\,dx \quad \text{and} \quad \int 3x^{-2}\ln(x^{-1})\,dx$$
   are good choices.

5. (10 pts) Does
$$\int_0^8 \frac{5+\sin(x)}{x(8+\cos(x))}\,dx$$
   converge or diverge? Demonstrate unequivocally that your answer is correct.

   We'll use the comparison test to show that the integral diverges. Since $\sin x$ is between -1 and 1, $5+\sin x$ is between 4 and 6. Likewise since $\cos x$ is between -1 and 1, $8+\cos x$ is between 7 and 9. It follows that
$$\frac{4}{9} \le \frac{5+\sin(x)}{8+\cos(x)} \le \frac{6}{7}$$
   for all values of x. Therefore
$$\int_0^8 \frac{5+\sin(x)}{x(8+\cos(x))}\,dx \ge \int_0^8 \frac{4}{9}\cdot\frac{1}{x}\,dx = \lim_{a\to 0^+}\frac{4}{9}\int_a^8 \frac{1}{x}\,dx = \lim_{a\to 0^+}\frac{4}{9}\ln x\Big|_a^8 = \frac{4}{9}\lim_{a\to 0^+}\ln(8)-\ln(a).$$
   Since $-\ln(a)$ approaches ∞ as a approaches 0, the final expression diverges. So the original integral diverges by the comparison test.

6. (10 pts) A common entry in many integral tables is:

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

Show how to use this information to calculate $\int \cos^4(x)\,dx$. For full credit, show all your work.

We'll need to apply the identity twice, once for $\cos^4$ and once for $\cos^2$, before we get to something we know how to do:

$$\begin{aligned}\int \cos^4(x)\,dx &= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx \\ &= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\left[\frac{1}{2}\cos x\sin x + \frac{1}{2}\int \cos^0 x\,dx\right].\end{aligned}$$

But $\cos^0 x = (\cos x)^0 = 1$, so the last integral is just x. Therefore we have

$$\int \cos^4(x)\,dx = \boxed{\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + C}.$$

Check by taking the derivative:

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}\cos x\sin x + \frac{3}{8}x\right) \\ &\quad= \frac{1}{4}\left(\cos^3 x(\cos x) + 3\cos^2 x(-\sin x)\sin x\right) + \frac{3}{8}\left(\cos x\cos x + (-\sin x)\sin x\right) + \frac{3}{8} \\ &\quad= \frac{1}{4}\left(\cos^4 x - 3\cos^2 x\sin^2 x\right) + \frac{3}{8}\left(\cos^2 x - \sin^2 x\right) + \frac{3}{8}\end{aligned}$$

Since we want to end up with cosines, change $\sin^2 x$ to $1 - \cos^2 x$:

$$\begin{aligned}&= \frac{1}{4}\cos^4 x - \frac{3}{4}\cos^2 x\left(1 - \cos^2 x\right) + \frac{3}{8}\cos^2 x - \frac{3}{8}\left(1 - \cos^2 x\right) + \frac{3}{8} \\ &= \frac{1}{4}\cos^4 x - \frac{3}{4}\cos^2 x + \frac{3}{4}\cos^4 x + \frac{3}{8}\cos^2 x - \frac{3}{8} + \frac{3}{8}\cos^2 x + \frac{3}{8} \\ &= \cos^4 x.\end{aligned}$$

Phew!

7. (10 pts) Using integration by parts, calculate

$$\int e^{-x}\cos(x)\,dx.$$

Let I be the integral in question. Then use integration by parts, with

$$\begin{aligned} u &= \ \ e^{-x} & v &= \sin(x) \\ u' &= -e^{-x} & v' &= \cos(x) \end{aligned}$$

to get

$$\begin{aligned} I &= \int uv'\,dx = uv - \int u'v\,dx = e^{-x}\sin(x) - \int -e^{-x}\sin(x)\,dx \\ &= e^{-x}\sin(x) + \int e^{-x}\sin(x)\,dx. \end{aligned}$$

Now do it again, with

$$\begin{aligned} u &= \ \ e^{-x} & v &= -\cos(x) \\ u' &= -e^{-x} & v' &= \ \ \sin(x) \end{aligned}$$

and get

$$\begin{aligned} I &= e^{-x}\sin(x) + \int uv'\,dx = e^{-x}\sin(x) + uv - \int u'v\,dx \\ &= e^{-x}\sin(x) + e^{-x}(-\cos(x)) - \int -e^{-x}(-\cos(x))\,dx \\ &= e^{-x}(\sin(x) - \cos(x)) - I. \end{aligned}$$

We got back where we started from, but with a minus sign, so we're OK. Now move the new I over to the other side and divide by 2 to get

$$I = \boxed{\frac{e^{-x}(\sin(x) - \cos(x))}{2} + C}.$$

Check by taking the derivative:

$$\begin{aligned} \frac{d}{dx}\left(\frac{e^{-x}(\sin(x) - \cos(x))}{2}\right) &= \frac{1}{2}\left(e^{-x}(\cos(x) - (-\sin(x))) + (-e^{-x})(\sin(x) - \cos(x))\right) \\ &= \frac{1}{2}e^{-x}(\cos(x) + \sin(x) - \sin(x) + \cos(x)) \\ &= e^{-x}\cos(x) \end{aligned}$$

as expected.

8. (10 pts) Calculate the exact value of this definite integral. You will be graded on the correctness of your work, so show it carefully.

$$\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx$$

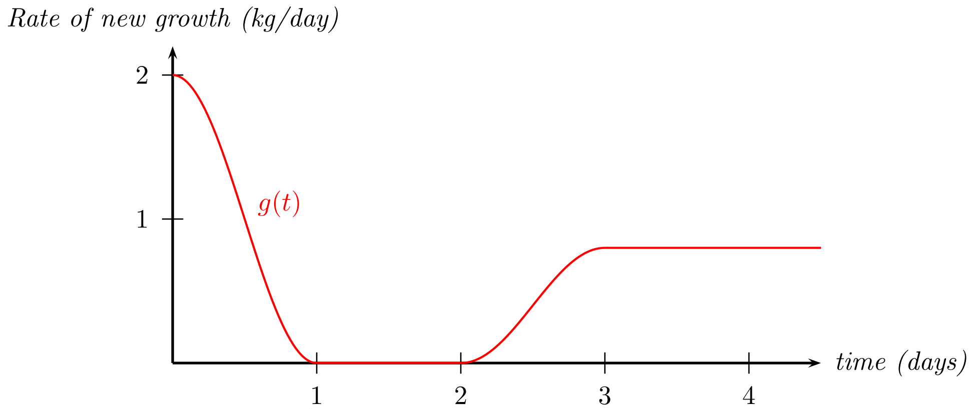
Let $w = 1 + e^x$. Then $dw = e^x dx$, so we have

$$\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx = \int_{x=-\infty}^{x=0} \frac{dw}{w} = \ln w\big|_{x=-\infty}^{x=0} = \ln(1+e^x)\big|_{-\infty}^{0}$$
$$= \lim_{a\to-\infty} \ln(1+e^x)\big|_{a}^{0} = \lim_{a\to-\infty} \ln(1+e^0) - \ln(1+e^a).$$

Since e^a approaches 0 as a approaches $-\infty$, that's

$$\ln(1+1) - \ln(1+0) = \boxed{\ln 2}.$$

9. (14 pts) In springtime, as a nameless old tree quietly builds its leaves and branches by drawing matter out of the air and earth, an unnamed old botanist measures the process with care:



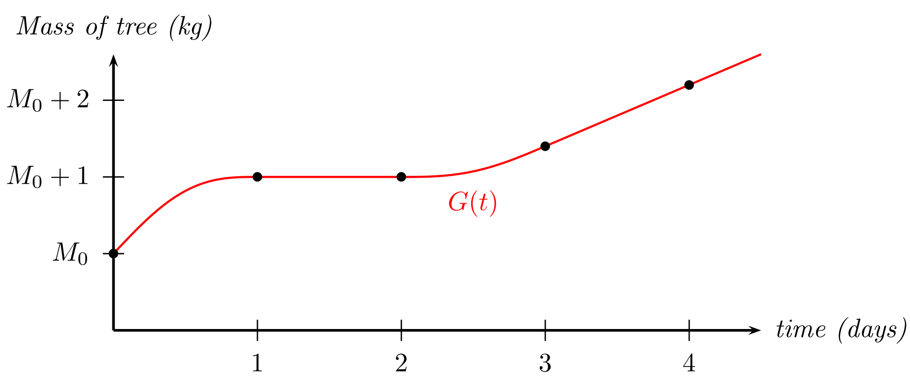
(a) (2 pts) Estimate $\int_0^4 g(t)\,dt$.

$$\text{TRAP}(4) = 1\left(\frac{1}{2}g(0) + g(1) + g(2) + g(3) + \frac{1}{2}g(4)\right) = \frac{1}{2}(2) + 0 + 0 + .8 + \frac{1}{2}(.8) = \boxed{2.2}.$$

(b) (4 pts) Explain what your answer to part (a) tells you about the tree.

The tree grew about 2.2 kg in the four days between time 0 and time 4.

(c) (6 pts) Sketch a possible graph of the mass of the tree as a function of time during this particular season. Label your graph carefully.



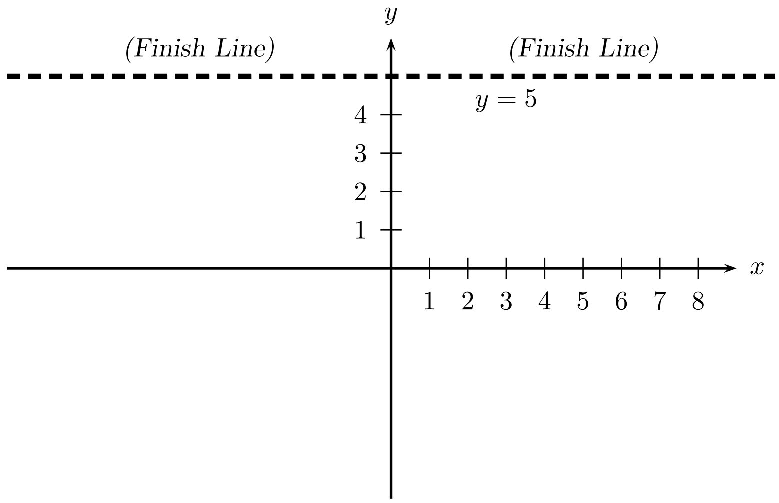
Here M_0 is the initial mass of the tree.

(d) (2 pts) What can you say about the derivative of the function you sketched in part (c)?

The derivative of $G(t)$ is $g(t)$.

10. The newest FOX reality show, "BattleBugs: Clash of the Beetles" begins (at $t = 0$) with eight assorted insects placed randomly on a large mat (the "battlefield", pictured here), on which is marked a "finish line". The producers hoped that the bugs would battle to be first to cross the finish line, but instead they wander around, each according to its nature. The motion of each bug is described by the equations below. Both x and y are measured in inches.

The Battlefield



Hercules Beetle $x(t) = \cos(t/2)$ $y(t) = \sin(t/2)$	**Ladybug** $x(t) = e^{-t}$ $y(t) = e^{-2t}$	**Tiger Beetle** $x(t) = 1 + t$ $y(t) = -1 + 8t$	**Longhorned Beetle** $x(t) = 3 + t$ $y(t) = 4 - t$
Dung Beetle $x(t) = t$ $y(t) = -2$	**Scarab** $x(t) = 2 - 7t$ $y(t) = -1 - 7t$	**June Beetle** $x(t) = 0$ $y(t) = -1$	**African Ground Beetle** $x(t) = \sin(t)$ $y(t) = \cos(t)$

Which bug (or bugs)...

(a) move repetitively?

The Hercules Beetle and African Ground Beetle move in circles. The others all move along lines, except for the ladybug.

(b) move fastest?

The Scarab Beetle moves fastest. Its velocity is $\sqrt{7^2 + 7^2} \approx 9.9$. This is faster than the Tiger Beetle, whose velocity is $\sqrt{1^2 + 8^2} \approx 8.1$.

(c) begin closest to the finish line?

$y(0)$ determines how close to the finish line a beetle starts. The Longhorned Beetle begins at $y = 4$, only one unit from victory. Unfortunately, it moves the wrong way.

(d) will reach the finish line first?

Most of the beetles either wander around or go in the wrong direction. The only one who actually moves consistently in the positive y direction (i.e., has $dy/dt > 0$) is the Tiger Beetle.

(e) will move very slowly (or not at all), in the long run?

The June Beetle doesn't move at all. The Ladybug gets slower and slower as she approaches the origin.