MATH 116 — EXAM II

DEPARTMENT OF MATHEMATICS
University of Michigan

March 18, 2003

NAME: SOLUTIONS ID NUMBER: ____________________

SIGNATURE: ____________________

INSTRUCTOR: ____________________ SECTION NO: ____________________

1. This exam has nine pages including this cover. There are nine questions.
2. Use of books, notes, or scratch paper is **NOT** allowed. You may certainly use your calculator (but not its manual). One 3x5-inch notecard is allowed.
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

PROBLEM	POINTS	SCORE
1	6	
2	8	
3	12	
4	10	
5	18	
6	10	
7	16	
8	10	
9	10	
TOTAL	100	

1. (2 pts each) Circle true or false. No explanation necessary.

True or $\boxed{\textbf{False}}$: The Taylor series centered at $x = 2$ of e^x is

$$1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \frac{(x-2)^4}{4!} + \frac{(x-2)^5}{5!} + \dots$$

This is e^{x-2}. If you multiplied all the terms by e^2, you'd have e^x centered at 2.

True or $\boxed{\textbf{False}}$: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges.
The harmonic series ($a_n = 1/n$) is a counterexample.

$\boxed{\textbf{True}}$ or **False**: The fourth derivative of

$$f(x) = 6 + 6x + 6x^2 + 6x^3 + \frac{x^4}{4} + 6x^5 + 6x^6 + 6x^7 + \dots$$

at the point $x = 0$ is 6.
The only term that matters is the x^4. After 4 derivatives it will be $(\frac{1}{4})(4)(3)(2)(1) = 6$.

2. (8 pts) Art inspires art: Whenever artists create art, other people have the opportunity to take inspiration and create more art. These responses can inspire still other works, and so on.

Assume that creating any number x of artistic works will inspire $.8x$ more artistic works (fractional works are ok–don't round). If you produce 75 sculptures, for how many total works can you claim indirect influence? In other words, how many artistic works are there, in total, which are part of your collection, inspired by your collection, inspired by those inspired by your collection, etc? Show your work carefully. Do not round, and do not estimate.

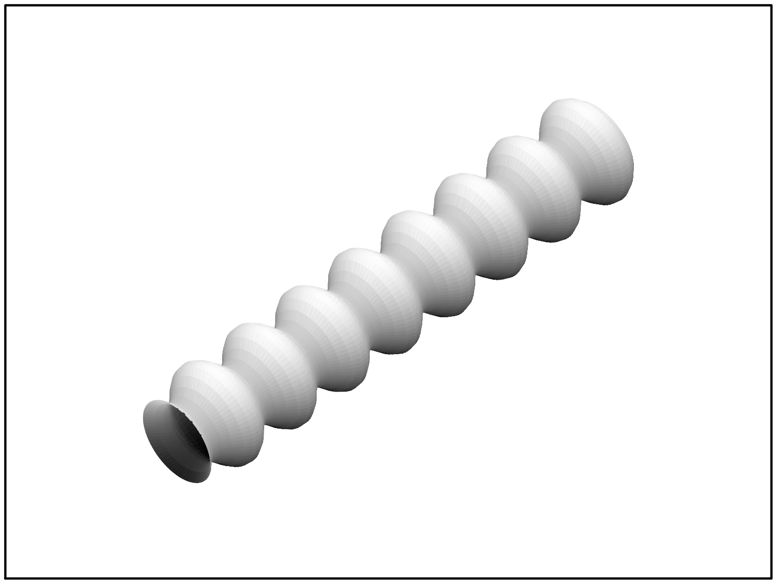
The works you can take credit for are:

Works you create	75
Works inspired by works you create	75(.8)
Works inspired by the works on the line above	75(.8)(.8)
Works inspired by the works on the line above	75(.8)(.8)(.8)
Etc.	

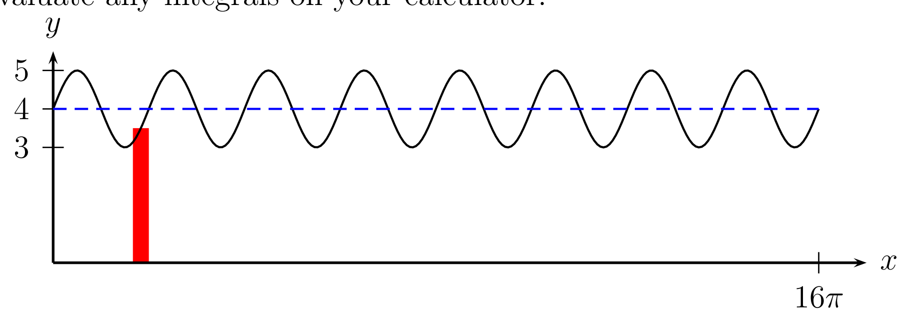
So the total is

$$75(1 + .8 + .8^2 + .8^3 + \cdots) = \frac{75}{1-.8} = \boxed{375 \text{ works of art}}.$$

3. (12 pts) A decorative table leg (see diagram) is manufactured so that it is the volume of rotation of the function $f(x) = 4 + \sin(x)$ between $x = 0$ and $x = 16\pi$.



a. (8 pts) What is the volume of the table leg? Show all work, but (on this problem only) you may evaluate any integrals on your calculator.



The red rectangle, when rotated around the x-axis, represents a slice of the chair leg. We have

$$\text{Volume of slice} = \pi(4 + \sin(x))^2 \Delta x$$

so

$$\text{Total volume} = \int_0^{16\pi} \pi(4 + \sin(x))^2 \, dx \approx 2606 \,\text{in}^2.$$

If you do the integral algebraically you'll find that the exact volume is $264\pi^2$.

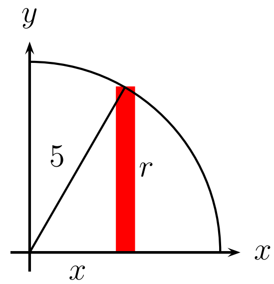
b. (4 pts) Here's a plausible shortcut: Replace the complicated shape with a cylinder with height 16π and radius 4 (because the average radius above is 4), and apply the volume formula for a cylinder. Is the shortcut valid? Explain briefly.

$\boxed{\text{No, not valid}}$. In this case, the shortcut would give an answer of

$$\pi r^2 h = \pi(4)^2(16\pi) = 256\pi^2 \approx 2527,$$

which we know to be too low. The peaks and valleys in the silhouette above correspond to bumps and grooves in the chair leg. The problem is that while the peaks and valleys are symmetric (i.e., they have the same area, so eliminating them both leaves the area of the sihouette the same), the bumps and grooves are not. The bumps have more volume than the grooves, because their slices have larger radii.

4. (10 pts) A sphere has uniform density ρ and radius 5. Of course, its center of mass is located at the center of the sphere. Assume now that the left half is cut off and removed. What is the center of mass of the remaining right hemisphere?



The hemisphere is the quarter circle above rotated about the x-axis. Clearly $\bar{y}$ and $\bar{z}$ are 0 by symmetry. To find $\bar{x}$, we can split the hemisphere up into slices like the one shown in red. By the Pythagorean Theorem $r = \sqrt{5^2 - x^2}$, so

$$\begin{aligned}
\text{Volume of slice} &= \pi r^2 \Delta x = \pi \left(25 - x^2\right) \Delta x \\
\text{Mass of slice} &= \rho(\text{Volume of Slice}) = \rho\pi \left(25 - x^2\right) \Delta x \\
\text{Moment of slice} &= x(\text{Mass of Slice}) = \rho\pi x \left(25 - x^2\right) \Delta x
\end{aligned}$$

so

$$\begin{aligned}
\text{Total Mass} &= \int_0^5 \rho\pi \left(25 - x^2\right)\, dx = \rho\pi \left(25x - \frac{1}{3}x^3\right)\Big|_0^5 = \frac{250\rho\pi}{3} \\
\text{Total Moment} &= \int_0^5 \rho\pi x \left(25 - x^2\right)\, dx = \rho\pi \int_0^5 \left(25x - x^3\right)\, dx \\
&= \rho\pi \left(\frac{25}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^5 = \frac{625\rho\pi}{4}.
\end{aligned}$$

Therefore

$$\bar{x} = \frac{\text{Total Moment}}{\text{Total Mass}} = \frac{625\rho\pi/4}{250\rho\pi/3} = \boxed{\frac{15}{8}}.$$

Notice that the density, ρ, turned out not to matter. That's because center of mass is a property of shape, not substance; a hempisphere made of gold has the same center of mass as a hemisphere made of styrofoam.

5. (18 pts) Fred likes to juggle. So does jason. The number of minutes Fred can juggle five balls without dropping one is a random variable, with probability density function $f(t) = 0.8e^{-0.8t}$. Similarly, the function $j(t) = 1.5e^{-1.5t}$ describes jason's skill. Here t is time *in minutes*.

a. (2 pts) Find $\int_0^\infty f(t)dt$. No need to show work.

That integral must be $\boxed{1}$, because f is a pdf.

b. (5 pts) What percentage of jason's juggling attempts are "embarrassing," meaning they last for 10 seconds or less? Show your work.

10 seconds is $\frac{1}{6}$ minute. So the proportion we want is

$$\int_0^{1/6} j(t)\,dt = \int_0^{1/6} 1.5e^{-1.5t}\,dt = -e^{-1.5t}\Big|_0^{1/6} = -e^{-1.5(1/6)} - (-e^0) = 1 - e^{-1/4} \approx \boxed{22\%}.$$

c. (6 pts) How long can Fred juggle, on average? Show your work.

$$\text{Fred's Average Time} = \int_0^\infty t f(t)\,dt = \int_0^\infty 0.8te^{-0.8t}\,dt.$$

Parts:

$$\begin{array}{ll} u = t & v = -e^{-0.8t} \\ u' = 1 & v' = 0.8e^{-0.8t} \end{array}$$

which yields:

$$\begin{aligned} \int 0.8te^{-0.8t}\,dt &= \int uv'\,dt = uv - \int u'v\,dt \\ &= -te^{-0.8t} - \int -e^{-0.8t}\,dt = -te^{-0.8t} - \frac{1}{0.8}e^{-0.8t} + C \\ &= -(t+1.25)e^{-0.8t} + C. \end{aligned}$$

Therefore

$$\begin{aligned} \text{Fred's Average Time} &= \lim_{b\to\infty} -(t+1.25)e^{-0.8t}\Big|_0^b \\ &= \left(\lim_{b\to\infty}\left(-(b+1.25)e^{-0.8b}\right)\right) - \left(-(0+1.25)e^{-0.8(0)}\right) \\ &= 1.25 - \lim_{b\to\infty}(b+1.25)e^{-0.8b}. \end{aligned}$$

Exponentials always beat polynomials, so $e^{-0.8b}$ goes to 0 faster than $b+1.25$ goes to infinity. So the last limit is 0, which makes Fred's average time $\boxed{\text{1.25 minutes}}$.

d. (5 pts) Who is the better juggler? Give a good reason for your decision.

Fred's average time was $1.25 = 1/0.8$, so Jason's will be $.67 = 1/1.5$. So $\boxed{\text{Fred}}$ has a better average.

6. (10 pts) Does $\frac{1}{3\ln(3)} + \frac{1}{4\ln(4)} + \frac{1}{5\ln(5)} + \frac{1}{6\ln(6)} + \ldots$ converge or diverge? Demonstrate unequivocally that your answer is correct.

Integral test:

$$\sum_{n=3}^{\infty} \frac{1}{n\ln(n)} \text{ converges if and only if } \int_3^{\infty} \frac{dx}{x\ln(x)} \text{ converges.}$$

So let's look at that integral. If we substitute $w = \ln(x)$, then $dw = dx/x$, so

$$\int \frac{dx}{x\ln(x)} = \int \frac{dw}{w} = \ln(w) + C = \ln(\ln(x)) + C.$$

That means

$$\int_3^{\infty} \frac{dx}{x\ln(x)} = \lim_{b\to\infty} \ln(\ln(x))\big|_3^b = \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(3)).$$

Now for large x, $\ln(x)$ is proportional to the number of digits in x. So it does go to infinity as x gets large, but very slowly. That means that $\ln(\ln(x))$ is like the number of digits in the number of digits in x. So it, too, goes to infinity, but really, *really* slowly. Nevertheless, the integral diverges, so $\boxed{\text{the sum diverges}}$ as well.

7. (16 pts)

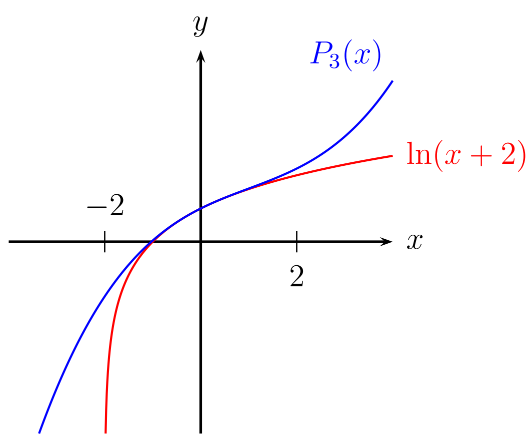
a. (8 pts) Find the Taylor series expansion for the function $\ln(2+x)$ centered at the point $x = 0$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(2+x)$	$\ln 2$
1	$(2+x)^{-1}$	2^{-1}
2	$(-1)(2+x)^{-2}$	$-1! \cdot 2^{-2}$
3	$(-2)(-1)(2+x)^{-3}$	$2! \cdot 2^{-3}$
4	$(-3)(-2)(-1)(2+x)^{-4}$	$-3! \cdot 2^{-4}$
n	$-(n-1)\cdots(-1)(2+x)^{-n}$	$(-1)^{n-1}(n-1)! \cdot 2^{-n}$

So the Taylor series is

$$\ln(2+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)! \cdot 2^{-n}}{n!} x^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 2^{-n} x^n}{n}$$
$$= \boxed{\ln 2 + \frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{3}\left(\frac{x}{2}\right)^3 - \frac{1}{4}\left(\frac{x}{2}\right)^4 + \cdots}$$

b. (4 pts) Using your calculator, graphically approximate the domain of convergence of this Taylor series. Accurately sketch a graph which suggests how you got your answer.



Graphing $\ln(x+2)$ together with the third-order Taylor polynomial

$$P_3(x) = \ln(2) + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24}$$

yields the picture on the left. It suggests that the domain of convergence is $(-2, 2)$, since that is the region on which the polynomial approximates the function well. Adding more terms to the polynomial gives a graph which fits more closely on $(-2, 2)$ and diverges more quickly elsewhere. So this is probably the domain of convergence.

c. (4 pts) It is claimed "One way to approximate $\ln(10)$ is to plug in 8 to the series above, using the first dozen, hundred, or even more terms. The more terms you take, the better your approximation of $\ln(10)$." Explain why this plan will (or will not) work.

That would work if 8 were in the domain of convergence of the Taylor series. But plugging 8 into the Taylor series will give a silly nonconvergent mess, and so that would not help calculate $\ln(10)$. We could, however, calculate $\ln(\pi)$ by plugging 1.14159 into the formula.

8. (10 pts) On this problem you must show your work and use exact methods. That is, calculator approximations are insufficient.

Find two values of x for which

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \frac{x^{22}}{11!} + \ldots = 1.$$

Despite the high number of points and the ugly formula, there's really only one idea: The formula above is the Taylor series for $\sin(x)$, except that the powers of x are twice as high as they should be. That means that the ugly expression above is in fact $\sin(x^2)$. Simplifying: $\sin(x^2) = 1$, so

$$x^2 \in \left\{\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots\right\}$$

which means

$$x \in \left\{\pm\sqrt{\frac{\pi}{2}}, \pm\sqrt{\frac{5\pi}{2}}, \pm\sqrt{\frac{9\pi}{2}}, \ldots\right\}.$$

9. (10 pts) You have been offered a deferred student loan: You will be paid $\$10,000$ today, but after four years elapse you must begin a continuous repayment stream at a rate of \$2000 per year for five years. Assume all moneys earn continuous 6% interest.

You have concocted the following (ethically questionable) plan:

1. Take the cash.
2. Put the cash in the bank, earning 6% continuous interest.
3. In four years, begin repayment, as scheduled. Continue repayment for five years.

Incidentally, why "ethically questionable"? If you can afford to save the money, then you obviously don't need the loan now. But student loans are usually need-based, especially deferred interest-free student loans (as this is). Is it unethical to accept a need-based loan when you don't need it? I think so. -jason

a. (4 pts) What is the future value ("future" means nine years later, when the entire loan is finally repaid) of the payment stream with which you pay back the loan?

Applying the future value formula:

$$\text{Future Value} = \int_4^9 P(t)e^{r(M-t)}\,dt = \int_4^9 2000e^{0.06(9-t)}\,dt = 11662.$$

That's $\boxed{\$11{,}662}$. It's more than the number of dollars paid, because they can earn interest after they're paid. Arguably, this number and the payment stream \$2000 should be negative numbers because they represent moneys paid out.

b. (4 pts) What is the future value of the \$10,000 received today?

This is much easier. It's

$$\$10,000e^{0.06(9)} = \boxed{\$17,160}.$$

That's how much money you'll have in nine years if you start at ten thousand and get 6% interest for nine years.

c. (2 pts) If you implement the plan above, how much "profit" will you have made in nine years?

The difference between the answers above, because one is the future value of money you get, and one is the future value of money you pay. So $\boxed{\$5{,}498}$. This can be regarded as the size (measured in future value) of the gift you have been given by being granted this super deal!