MATH 116 — Final Exam

DEPARTMENT OF MATHEMATICS
University of Michigan

April 21, 2003

NAME: ______________________________ ID NUMBER: ____________________________

SIGNATURE: _______________________

INSTRUCTOR: ____________________ SECTION NO: ____________________________

1. This exam has eleven pages including this cover. There are eleven questions.
2. Use of books, notes, or scratch paper is **NOT** allowed. You may certainly use your calculator (but not its manual). One 3x5-inch notecard is allowed.
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

PROBLEM	POINTS	SCORE
1	10	10
2	10	10
3	13	13
4	13	13
5	6	6
6	6	6
7	9	9
8	8	8
9	6	6
10	11	11
11	8	8
TOTAL	100	100

1. (2 pts each) Circle true or false. No explanation necessary.

True or **False**: If f is a continuous function, then $\int_0^9 f(x)dx$ is between LEFT(17) and RIGHT(17).
FALSE. LEFT(17) and RIGHT(17) could both be equal and different from the integral. This will happen, for example, if $f(0) = f(9)$ and f is concave up.

True or **False**: The sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ diverges.
TRUE. This is the harmonic series.

True or **False**: Applying separation of variables to a differential equation is always more accurate than using Euler's method.
FALSE. This problem was spoiled by misphrasing. It should have said "is always *at least* as accurate as Euler's method," which is true (whenever separation is possible). It can (rarely) happen that Euler's method is exact for a particular function, in which case separation is no *more* accurate. Credit was given for both answers to this problem.

True or **False**: Let $p(x)$ be a probability density function. Then for all x, $0 \leq p(x) \leq 1$.
FALSE. The cumulative distribution function would have to be so bounded, but not the density function.

True or **False**: If a metal rod has variable density $\rho(x)$ kilograms per meter, then its mass is the product of its length and the integral of $\rho(x)$.
FALSE. No need to multiply by the length after integrating $\rho(x)$.

2. (10 pts)

a. (5 pts) Write a single differential equation, all of whose solutions approach the value 60 in the long run.

$$\frac{dy}{dt} = -2(y - 60)$$

Note that if $y > 60$, $\frac{dy}{dt}$ will be negative, while if $y < 60$, $\frac{dy}{dt}$ will be positive. So the equilibrium $y = 60$ is stable.

b. (1 pt) Write another differential equation with the same property.

$$\frac{dy}{dt} = -3(y - 60)$$

c. (4 pts) Write a general solution to your differential equation from part a.

$$y(t) = 60 + De^{-3t}$$

3. (13 pts) The Altairian slime mold's rate of growth is inversely proportional to its weight. A culture which four days ago weighed only 3 grams has already grown to 5 grams.

a. (10 pts) Using a differential equation, find a formula for the slime mold's weight as a function of time. **Show all of your work.**

$$\begin{aligned} \frac{dw}{dt} &= \frac{k}{w} \\ wdw &= kdt \\ \int wdw &= \int kdt \\ \frac{w^2}{2} &= kt + C \\ w^2 &= 2kt + 2C \\ w &= \sqrt{2kt + 2C} \end{aligned}$$

Notice that $w(0) = 5$ and $w(-4) = 3$, so...

$$\begin{aligned} 5 &= \sqrt{0 + 2C} \\ 12.5 &= C \\ 3 &= \sqrt{2k(-4) + 25} \\ 2 &= k \\ w &= \sqrt{4t + 25} \end{aligned}$$

b. (3 pts) Calculate the slime mold's age. (It begins its life as a microscopic spore.)

When the slime mold begins its life, its weight is essentially 0, so we set $0 = \sqrt{4t + 25}$. Solving for t gives $t = -6.25$. Because (according to the choices above) $t = 0$ corresponds to the present, this means the slime mold is 6.25 days old.

Trivia about this problem: "Altair" is the name of the planet which is the setting of the 1956 classic SciFi movie "Forbidden planet," starring, among others, Robby the Robot. To the best of my knowledge, there are no slime molds in the movie. The 3-4-5 right triangle convenience, namely that $3^2 + 4^2 = 5^2$ is used in this problem to make the numbers turn out nicely.

4. (13 pts) In 1970 the worldwide black rhinoceros population was approximately 65,000. Assume that until 1970 the population was at the natural equilibrium population of this logistic differential equation:

$$dR/dt = .12R - \frac{R^2}{540,000}$$

But around 1970 demand for rhinoceros horn[1] dramatically increased poaching...

a. (4 pts) A scientist claims "Each year poachers now take 10% of the population!" If this is true, how should the differential equation be changed to reflect this fact? *Is it still logistic?*

10% of the populatino is $.1R$. So this rate needs to be subtracted from the formula above for dR/dt. That gives

$$dR/dt = .02R - \frac{R^2}{540,000}.$$

This is still logistic, because it's still a quadratic in R with no constant term.

b. (3 pts) If the scientist is correct, what will happen to the rhino population in the long run? No explanation necessary, but answer clearly.

Because it's still logistic, we need only find the stable carrying capacity. There are lots of ways to do this, but perhaps the easiest is to find all equilibria by setting $.02R - \frac{R^2}{540,000}$ equal to zero and solving for R. That produces two solutions, namely $R = 0$ and $R = 540,000 * 0.02 = 10800$. In the long run, the rhino population will approach 10800 members.

c. (6 pts) As of 2003, there are approximately 2,500 black rhinos left. If poaching can be eradicated, so that the population again follows the original logistic model, when will the population reach 50,000? Show all work.

If poaching is eradicated, it will follow the original equation above. Putting that into the more convenient logistic form gives:

$$dR/dt = .12R(1 - \frac{R}{64,800})$$

So the solution curve will look like:

$$R(t) = \frac{64,800}{1 - Ae^{-.12t}}, \text{ where } A = (64800 - 2500)/2500 = 24.92.$$

Solving for t gives $t = 37$ years.

Factoids: The Rhino populations given for 1970 and 2003 are true–the world saw a 95% drop in this population over this brief period. Illegal poaching has indeed been the primary cause.

[1] for dagger handles in North Yemen and folk medicine in Asia

5. (6 pts) Find all equilibrium solutions of the equation

$$dy/dt = 3t^2(y-4).$$

For each equilibrium, tell whether it is stable or unstable.

Set $3t^2(y-4) = 0$ and solve, producing the solutions $t = 0$ and $y = 4$. But a value of t cannot be an equilibrium solution–that wouldn't make sense. So $y = 4$ is the only equilibrium. Furthermore, dy/dt is positive when $y > 4$ and negative when $y < 4$, so the equilibrium is unstable.

6. (6 pts) Evaluate:

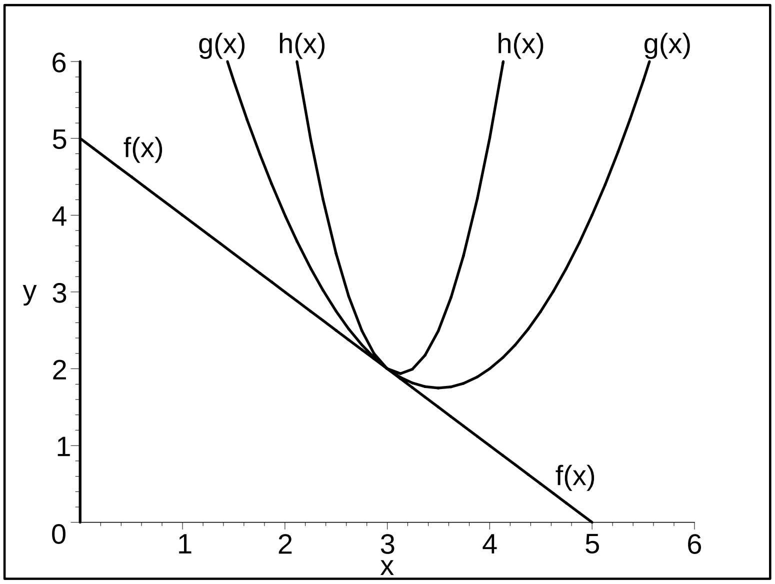
$$\int_{-1}^{2} \pi r e^{-3r} dr.$$

This "gateway problem" can be solved by integration by parts. Set $u = r$, $v' = e^{-3r}$, $u' = 1$, $v = \frac{-e^{-3r}}{3}$.

$$\int_{-1}^{2} \pi r e^{-3r} dr = \pi[-r\frac{e^{-3r}}{3}|_{-1}^{2} - \int_{-1}^{2} -\frac{-e^{-3r}}{3} dr] =$$

$$= \pi[-r\frac{e^{-3r}}{3} - \frac{-e^{-3r}}{9}]_{-1}^{2} = -14.028$$

7. (9 pts) Three functions $f(x)$, $g(x)$, and $h(x)$ are graphed below. Propose possible Taylor series centered at $x = 3$ for these three functions. It is important that your formulas for the three be compatible with each other. Include enough terms to distinguish between the functions.



$f(x) = 2 - (x - 3)$

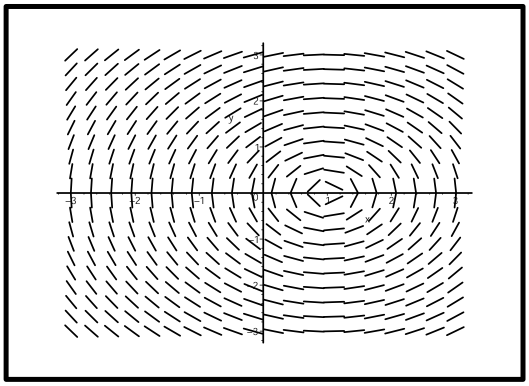
$g(x) = 2 - (x - 3) + (x - 3)^2$

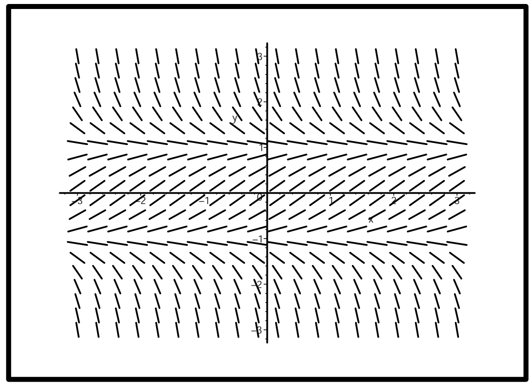
$h(x) = 2 - (x - 3) + 4(x - 3)^2$

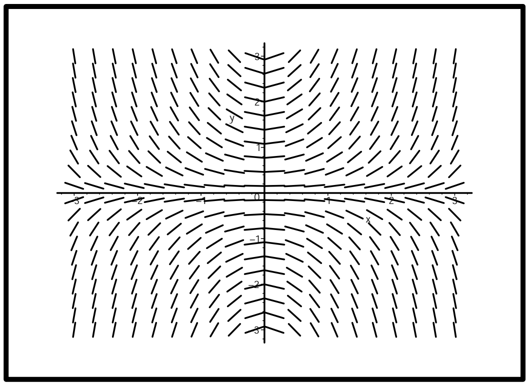
8. (Matching, 2 pts each) For each of the slope fields below, find its equation from the list of differential equations, and place the letter of its equation beside it.

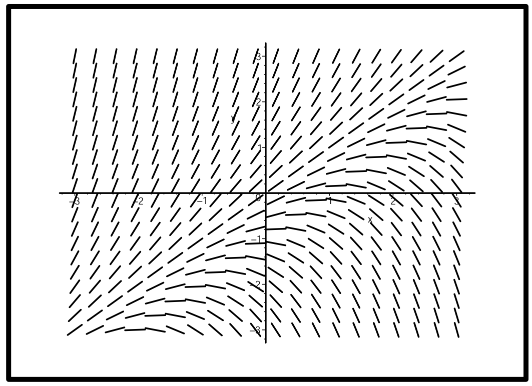
A: $\frac{dy}{dx} = y(1 - \frac{y}{2}) - 1$ C: $\frac{dy}{dx} = x - 1 - y$ E: $\frac{dy}{dx} = \frac{1-x}{y}$ G: $\frac{dy}{dx} = (x-1)^2 + y^2$

B: $\frac{dy}{dx} = x^3$ D: $\frac{dy}{dx} = 1 - x + y$ F: $\frac{dy}{dx} = xy$ H: $\frac{dy}{dx} = 1 - y^2$









Answers: E,H,F,D (in this order: upper left, upper right, lower left, lower right)
"Decoys": G,A,B,C (in the same order)

9. (6 pts) The chambered nautilus builds a spiral sequence of closed chambers. It constructs them from the inside out, with each chamber approximately 20% larger (by volume) than the last. (The large open section at the top is not a "chamber.") The largest chamber is 9 cubic inches. How much volume is enclosed by all the chambers? Assume for simplicity that there are infinitely many chambers. Show your work.

Because the chambers grow by a constant factor each time, they form a geometric series. If each is 20% larger than the previous, then the ratio between them is 1.2. But this is the ratio of the larger divided by the smaller, and we want the opposite, so we get $r = 1/1.2 = 5/6$. This is the ratio by which you have to multiply each volume to get the next smaller volume. The total volume, then, is:

$$9 + 9(\frac{5}{6}) + 9(\frac{5}{6})^2 + 9(\frac{5}{6})^3 \ldots.$$

This geometric series sums to $\frac{9}{1-\frac{5}{6}} = 54$. So the total enclosed volume is 54 cubic inches.

By the way, the numbers given in this problem are not simply made up, but are deduced from the size and shape of a large adult chambered nautilus. The number 54 is the approximate volume of a cylinder with height 2 inches and radius 3 inches (a rough approximation to the organism's size and shape).

Where does $5/6$ come from? Notice that one "band" of the chambers takes about 17 chambers, and (by directly measuring the picture), shrinks the organism by a factor of 3, *in length*. Scaling down by a factor of 3 in length is the same as scaling by a factor of 27 in volume, which should leave $54/27 = 2$ cubic inches. Therefore the first 17 chambers take 52 cubic inches. So we have the equations:

$$\frac{a}{1-r} = 54 \text{ and } \frac{a(1-r^{17})}{1-r} = 52.$$

Solving simultaneously gives $a = 9.5 \approx 9$, $r = .82 \approx 5/6$. This is how the problem was written.

10. (11 pts) The Whistling Thorn Acacia tree and a certain species of stinging ant closely share the same environment. The trees harbor and feed the ants, which, in turn, protect the trees from mammals and other insects. Assume:

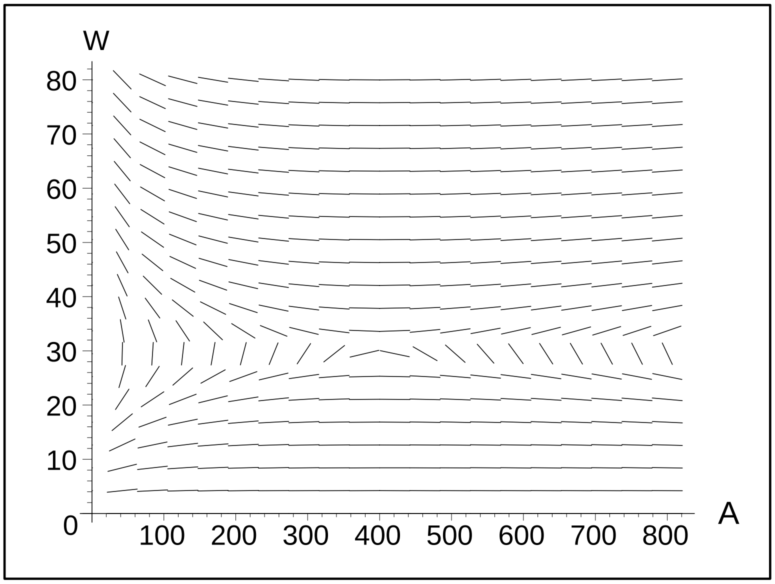
1. The yearly birth rate of the ant population $A(t)$ is proportional to the product of the number of ants and number of trees, with constant of proportionality 10^{-2}.
2. The **relative** yearly death rate of the ants is 30%.
3. The yearly birth rate of the tree population $W(t)$ is proportional to the product of the number of ants and the number of trees, with constant of proportionality 10^{-4}.
4. The **relative** yearly death rate of the trees is 4%.

a. (5 pts) Write a system of differential equations for these populations. Use variables A and W for the populations. No explanation necessary.

$$\frac{dA}{dt} = 10^{-2}AW - .3A$$
$$\frac{dW}{dt} = 10^{-4}AW - .04W$$

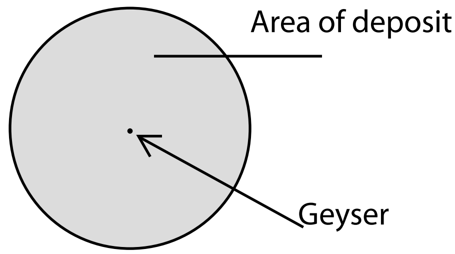
b. (1 pt) In your response to part a., circle the term which best expresses the truth "ants help trees." No explanation necessary. The term which is expresses this is the $10^{-4}AW$, which appears in the differential equation describing the trees.

c. (5 pts) Here is a picture of the phase plane for the system. At time $t = 0$, $A = 50$ and $W = 40$. Give an accurate graph of each $A(t)$ and $W(t)$. No explanation necessary. Use the back if you prefer.



Two graphs should be given on two new axes. The first, $A(t)$ should be labeled with the point $(0, 50)$, and should increase, then decrease to zero. The second, $W(t)$ should start at $(0, 40)$, and should simply decrease to zero.

11. (8 pts) The area surrounding a geyser is found to have high quantities of mineral deposits. Assume that at any radius $r < 5$ meters, $5 - r$ grams per square meter of the mineral silica can be found. How much silica is there, in total, within 5 meters of the geyser? Show your work.



This is a clone of the individual homework problem 8.3.11. Just as in that problem, you have to slice the circular region into strips which can be regarded as rectangles with length $2\pi r$ and width dr:

There are $\int_0^5 (5 - r)2\pi r dr = 5^3\pi/3 \approx 131$ grams of silica deposit