MATH 116 — FIRST MIDTERM EXAM

Solutions

Winter 2004

NAME: ____________________________ ID NUMBER: ____________________________

INSTRUCTOR: ____________________ SECTION NO: ____________________________

1. Do not open this exam until you are told to begin.
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	14	
2	10	
3	12	
4	15	
5	8	
6	10	
7	15	
8	16	
TOTAL	100	

1. (14 pts.) Suppose that f and its derivative f' are continuous functions such that $f(0) = -1$, $f(2) = 3$, $f'(0) = 3$, $f'(2) = 4$, and $\int_0^2 f(x)\,dx = 1.5$. Compute each of the following definite or indefinite integrals. Be sure to show your work.

(a) $\int f'(x)e^{2f(x)}\,dx$

$= \frac{1}{2}e^{2f(x)} + C$

because $\frac{d}{dx}\{\frac{1}{2}e^{2f(x)} + C\} = \frac{1}{2}e^{2f(x)} \cdot 2f'(x)$ **by the chain rule.**

(b) $\int_0^1 f(2x)\,dx$

$= \int_0^2 f(w)\,\frac{dw}{2}$ **by** $w = 2x$**,** $dw = 2dx$**,** $x = 0, x = 1 \implies w = 0, w = 2$

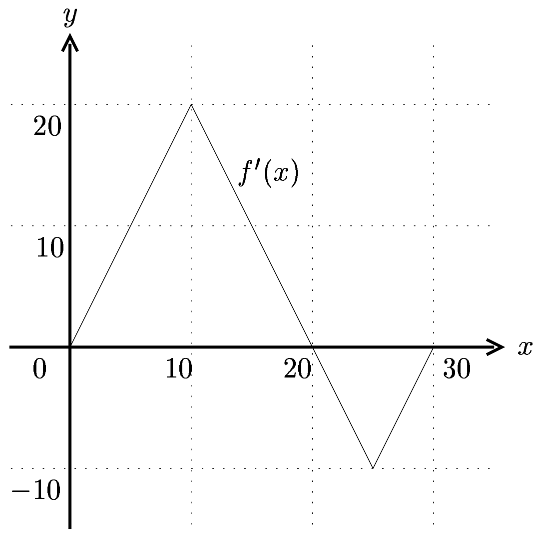
$= \frac{1}{2}\int_0^2 f(w)\,dw = \frac{1.5}{2} = 0.75$

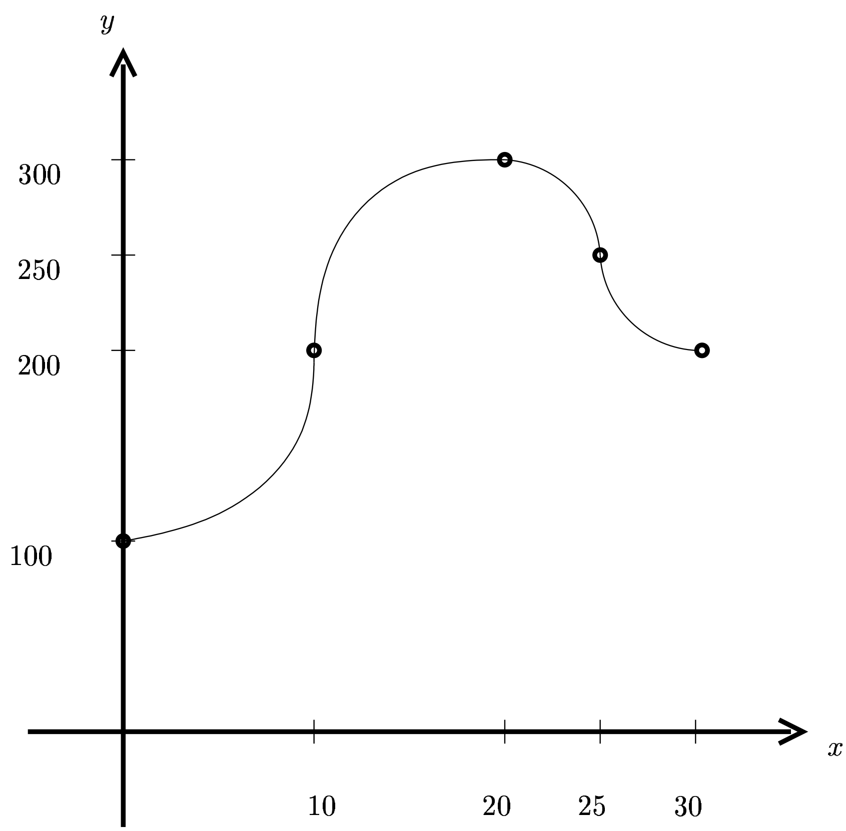
(c) $\int_0^2 xf''(x)\,dx$

$= xf'(x)|_0^2 - \int_0^2 f'(x)\,dx$ **using** $u = x, du = dx, dv = f''(x)dx, v = f'(x)$

$= 2 \cdot 4 - 0 - 3 + (-1) = 8 - 4 = 4$

2. (10 pts.) The graph of *the derivative*, $f'(x)$, of a function, $f(x)$, is given below. On the axes provided, sketch a graph of $f(x)$ provided that $f(0) = 100$. Be sure that your graph is labeled with appropriate units and that it shows clearly the main features of f such as local maxima and minima and inflection points.





3. (12 pts.) Some distance upriver from a small reservoir, there has been a chemical spill. The authorities are concerned with levels of the chemical in the reservoir. Consequently, they take samples at half hour intervals of the rate $r(t)$, in gallons per hour, that the chemical is entering the reservoir t hours after the chemical spill. Their data is recorded in the following table.

t	0	.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$r(t)$	0	0	0	0	0	0.175	.4	.675	1	1.375	1.8

(a) Write an integral that represents the total amount of chemical that has entered the reservoir during the first five hours after the spill.

$\int_0^5 r(t)\,dt$

(b) Based on the data given, find the left and right hand sum approximations to your integral. (Show how you computed the sums.)

LHS: $0.5(0+0+0+0+0+0.175+0.4+0.675+1+1.375) = 1.8125$
RHS: $0.5(0+0+0+0+0.175+0.4+0.675+1+1.375+1.8) = 2.7125$

LHS = 1.8125 gallons

RHS = 2.7125 gallons

(c) Is it reasonable to expect that LHS is a lower bound for the integral? Explain why or why not.

It is reasonable to expect that LHS is a lower bound for the integral because 1) we can see that $r(t)$ appears to be an increasing, concave-up function by calculating the difference quotients from the above table,
2) LHS is a lower bound for increasing functions

(d) What is your best estimate of the integral based on the given data? Do you think it would be an under- or over-estimate of the actual value of the integral? Explain the reason for your answer.

Since the left hand sum is an underestimate and the right hand sum is an over-estimate of the integral, their average, the trapezoidal approximation, will likely be a a better estimate of the integral. In general, we expect the trapezoidal estimate, in this case $TRAP = (LHS + RHS)/2 = 2.2625$, to give a better estimate because the lines given by taking the tops of the trapezoids usually give a better approximation to the function $r(t)$ than do the horizontal line seqments that are the tops of the rectangles in the left and right hand sums.
We expect the trapezoidal rule to be an overestimate since $r(t)$ appears to be concave up (r seems to be increasing at an increasing rate), so the tops of the trapezoids used to approximate the area under the graph of r lie above the graph.

4. (15 pts.) For each of the following statements about a continuous function, f, circle **T** if the statement is always true, and otherwise circle **F**. If a statement is always true, explain why. If a statement is not always true, give an example of a function so that the statement is not true for that function.

(a) $\int xf(x)\,dx = x\int f(x)\,dx.$ **T** [**F**]

The function $f(x) = x$ is a counterexample to the statement. For, if $f(x) = x$, the left hand side is $\frac{x^3}{3} + C$ while the right hand side is $x(\frac{x^2}{2} + C_1) \neq \frac{x^3}{3} + C$.

(b) Every function, $f(x)$, that is continuous on an interval, $[a, b]$, has an antiderivative on that interval. [**T**] **F**

This is a consequence of the *Construction Theorem for Antiderivatives*, Theorem 6.2 on page 279 of the text. By that theorem, an antiderivative for f is the function $F(x) = \int_a^x f(t)\,dt$.

(c) If f is a positive continuous function for $x \geq 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ converges. **T** [**F**]

The function $f(x) = 1/x$ is a counterexample to the statement. If $f(x) = \frac{1}{x}$, then $\lim_{x\to\infty} f(x) = 0$ but $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(b)$ which is not a finite number. So the integral diverges.

5. (8 points). If F is the function defined for $x > 0$ by $F(x) = \int_1^x \frac{e^t}{t}\,dt$, show that $\int F(x)\,dx = xF(x) - e^x + C.$

We want to show the right hand side is an antiderivative for F; that is $\frac{d}{dx}(xF(x) - e^x + C) = F(x)$.

However, $\frac{d}{dx}(xF(x) - e^x + C) = F(x) + xF'(x) - e^x$ by the product rule.

$F'(x) = \frac{e^x}{x}$ by the F.T.C.

So $\frac{d}{dx}(xF(x) - e^x + C) = F(x) + x\frac{e^x}{x} - e^x = F(x)$ as required.

6. (10 points) **(a)** Explain why $\int_0^{\pi/3} \frac{\cos x}{\sin^2 x}\, dx$ is an improper integral.

Because the integrand, $\frac{\cos(x)}{\sin^2(x)}$, is unbounded for x near 0.

(b) Show how the improper integral in part (a) is defined mathematicially.

$\int_0^{\pi/3} \frac{\cos x}{\sin^2 x}\, dx = \lim_{a\to 0^+} \int_a^{\pi/3} \frac{\cos x}{\sin^2 x}\, dx$ **provided the limit is a finite number.**

Otherwise, the integral diverges.

(c) If the improper integral in part **(a)** converges, then use your definition to calculate the value to which it converges. If the improper integral in part **(a)** does not converge, then explain why not. Show your work.

$\int_a^{\frac{\pi}{3}} \frac{\cos x}{\sin^2 x}\, dx = \int_{\sin(a)}^{\sin(\frac{\pi}{3})} \frac{1}{u^2}\, du = -\frac{1}{u}\Big|_{\sin(a)}^{\sin(\frac{\pi}{3})} = -\frac{1}{\sin(\frac{\pi}{3})} + \frac{1}{\sin(a)}$

by $u = \sin(x)$**,** $du = \cos(x)dx$**.**

$\lim_{a\to 0^+} \int_a^{\frac{\pi}{3}} \frac{\cos x}{\sin^2 x}\, dx = -\frac{1}{\sin(\frac{\pi}{3})} + \lim_{a\to 0^+} \frac{1}{\sin(a)}$ **and the last limit does not exist since** $\sin(a) \to 0$ **as** $a \to 0^+$**. Therefore, the integral diverges.**

7. (15 points) Let g be the function that is defined for $x > 1$ by

$$g(x) = \int_3^x \frac{t}{\ln t}\, dt.$$

(a) Find $g'(x)$.

$g'(x) = \dfrac{x}{\ln(x)}$

(b) On which subinterval(s) of $x > 1$, if any, is g increasing? Briefly explain the reason for your answer.

g is increasing where $g' > 0$, or where $\dfrac{x}{\ln(x)} > 0$. Thus g is increasing on every subinterval of $x > 1$.

(c) On which subinterval(s) of $x > 1$, if any, is g concave up? Briefly explain the reason for your answer.

g is concave up where g' is increasing or where $g'' > 0$. Now, $g''(x) = \dfrac{\ln(x) - 1}{(\ln(x))^2}$. Since the denominator is always positive, this expresssion is zero at $x = e$, positive for $x > e$ where $\ln x > 1$, and negative for $1 < x < e$ where $\ln x - 1 < 0$. Therefore, $g''(x) > 0$ and g is concave up for $x > e$.

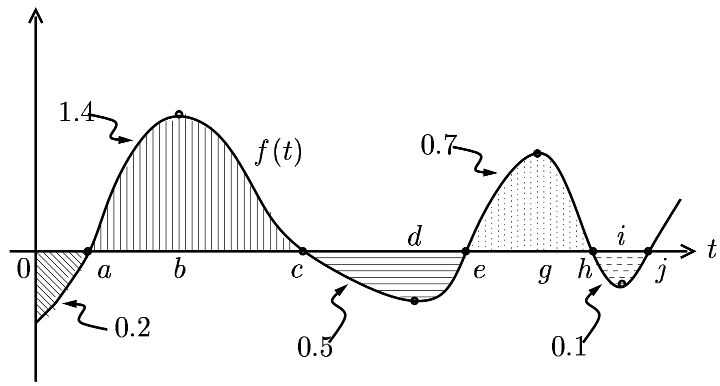
(d) Fill in the blanks with one of the words "positive", "negative", or "zero", to make the following sentences true. (Any word may be used more than once. No explanation necessary.)

$g(4)$ is positive.

$g(3)$ is zero.

$g(2)$ is negative.

8. (16 pts.) The function $f(t)$ represents the velocity (in meters per second) of a charged particle in a variable electromagnetic field, t seconds after the beginning of an experiment. Positive velocity represents travel away from the positively charged plate used in the experiment. The graph of f is shown below. The areas between the graph of f and the horizontal axis are also indicated.



(a) In the context of the question, briefly explain the meaning of the integral $\int_c^h f(t)\,dt$.

The particle is $\int_c^h f(t)\,dt$ meters farther from the plate at time $t = h$ than it is at time $t = c$.

(b) At which time(s) between $t = 0$ and $t = j$ is the particle furthest from the positively charged plate? How do you know this?

The particle is furthest from the positively charged plate at time $t = h$. This is because the particle is $\int_0^t f(s)\,ds$ meters farther from the plate at time t than it is at time 0. The function g is decreasing for $0 < t < a$, for $c < t < e$, and for $h < t < j$, and it is increasing for $a < t < c$ and $e < t < h$. Thus, g must be largest at either $t = 0$, $t = c$, or $t = h$. However, from the given graph and areas, $g(h) = g(c) - .5 + .7 = g(c) + .2 > g(c) = g(0) - .2 + 1.4 = g(0) + 1.2 > g(0)$. Therefore, for times between $t = 0$ and $t = j$, g is largest at $t = c$.

(c) What is the distance between the position of the particle at time $t = 0$ and its position at time $t = e$. Be sure to show the calculations used to obtain your answer.

0.7 meters. To find this we calculate $\displaystyle\int_0^a f(t)\,dt + \int_a^c f(t)\,dt + \int_c^e f(t)\,dt$ and note that the graph tells us that $\displaystyle\int_0^a f(t)\,dt = -0.2$, $\displaystyle\int_a^c f(t)\,dt = 1.4$, and $\displaystyle\int_c^e f(t)\,dt = -0.5$. Thus we obtain $(-0.2) + 1.4 + (-0.5) = 0.7$.

(d) What is the total distance travelled by the particle in the first e seconds? Be sure to show your calculation.

2.1 meters. To find this we calculate $\displaystyle\int_0^a |f(t)|\,dt + \int_a^c |f(t)|\,dt + \int_c^e |f(t)|\,dt$ and note that the graph tells us that $\displaystyle\int_0^a |f(t)|\,dt = 0.2$, $\displaystyle\int_a^c |f(t)|\,dt = 1.4$, and $\displaystyle\int_c^e |f(t)|\,dt = 0.5$. Thus we obtain $0.2 + 1.4 + 0.5 = 2.1$. (Note that we use $\int_p^q |f(t)|\,dt$ because we are concerned with distance travelled rather than change in displacement.)