MATH 116 — SECOND MIDTERM EXAM

Winter 2004

NAME: ______________________ ID NUMBER: ______________________

INSTRUCTOR: ________________ SECTION NO: ______________________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 11 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	4	
2	4	
3	8	
4	8	
5	8	
6	12	
7	9	
8	15	
9	10	
10	10	
11	12	
TOTAL	100	

1. (4 points) Find the sum of the infinite series

$$2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots+\left(\frac{2}{3}\right)^n+\ldots.$$

$$2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots+\left(\frac{2}{3}\right)^n+\ldots=2+\left(\frac{2}{3}\right)^2\left[1+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots+\left(\frac{2}{3}\right)^n+\ldots\right]$$

$$=2+\left(\frac{2}{3}\right)^2\left[\frac{1}{1-\frac{2}{3}}\right]=2+\frac{4}{3}=\frac{10}{3}.$$

2. (4 points) Does the infinite series $\sum_{n=1}^{\infty}\frac{n^3}{n^5+1}$ converge? Explain why or why not.

The infinite series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and the given infinite series has smaller positive terms because $0<\frac{n^3}{n^5+1}\leq\frac{n^3}{n^5}=\frac{1}{n^2}$. Therefore, the series converges by the comparison test.

3. (8 points) If the fourth degree Taylor polynomial approximating a function f near $x=-1$ is $P_4(x)=2-3(x+1)-(x+1)^3+4(x+1)^4$, then

(a) The linear approximation to f near $x=-1$ is $\boxed{2-3(x+1)}$.

(b) $f'''(-1)=$ $\boxed{-6}$.

4. (8 points) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Let $S_n = a_1 + a_2 + \cdots + a_n$, so that $S_n - S_{n-1} = a_n$. The series converges to S if and only if $S = \lim_{n\to\infty} S_n$ exists. In this case, $\lim_{n\to\infty} S_{n-1} = S$ as well so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

5. (8 points) In this question we will investigate the convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2}{e^n}(x+2)^n$.

(a) Find the radius of convergence, R, of the power series. (Show your work.)

Solution. We will use the ratio test. Let $a_n = \dfrac{n^2(x+2)^n}{e^n}$ so that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2(x+2)^{(n+1)}}{e^n}}{\frac{n^2(x+2)^n}{e^n}} = \frac{(n+1)^2(x+2)}{en^2}.$$

Consequently,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{en^2}|x+2| = \frac{|x+2|}{e}.$$

By the ratio test, the series therefore converges when this limit is less than one and diverges when it is greater than one. That is, the series converges for $|x+2| < e$ and diverges for $|x+2| > e$, so the radius of convergence is $R = e$. The series converges for $-e < x+2 < e$ or $-2-e < x < -2+e$.

$R =$ ___ e ___ .

(b) What is the interval of convergence of the power series?

$-2 - e < \quad x \quad < -2 + e.$

6. (12 points) A team of biologists is interested in the ability of certain birds to migrate great distances with little rest. The biologists are monitoring a flock of birds known to migrate after spending the winter in the warm climes of the Okefenokee swamp. The location of the flock, in $x(t)$ hundreds of miles north, and $y(t)$ hundreds of miles east of the base camp of the biologists, t days after their departure from the Okefenokee swamp is given by

$$x(t) = 3t + \frac{1}{2},$$
$$y(t) = t^{\frac{3}{2}} + \frac{1}{5}.$$

(a) Where is the Okefenokee swamp in relation to the base camp of the biologists?

At $t = 0$, the birds are at the swamp, so it is located $x(0) = .5$ hundred miles north and $y(0) = .2$ hundred miles east of the base camp of the biologists. That is, 50 miles north and 20 miles east.

(b) Is there ever a time when the flock of birds is travelling due North-East? If so, when? If not, explain why not.

The flock is travelling north-east at time t if $x'(t) = y'(t) > 0$. Since $x'(t) = 3$ and $y'(t) = \frac{3}{2}t^{1/2}$, this occurs when $3 = \frac{3}{2}t^{1/2}$ or when $t = 4$ days.

(c) Is the flock of birds constantly moving throughout the first three days of their journey? Why or why not?

The velocity of the flock at time t is $v(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{9 + \frac{9}{4}t} \geq 3 > 0$, so the flock is constantly in motion during all the days it is travelling.

(d) How far does the flock of birds travel in the first three days of their journey?

Since the rate of change of distance travelled with respect to time is the velocity, the distance travelled during the first three days is given by the integral.

$$\int_0^3 v(t)\,dt = \int_0^3 \sqrt{9 + \frac{9}{4}t}\,dt$$

One can calculate this integral, either by using the fact that an antiderivative for the integrand is $8(1 + \frac{t}{4})^{3/2}$ and applying the fundamental theorem of calculus, or by using numerical integration. The distance travelled is $(\sqrt{7})^3 - 8 \approx 10.52$ hundred miles, or about 1052 miles.

7. (9 points) A thin metal rod that is three meters long has density $\delta(x) = 1+kx^2$ kilograms/meter at a point that is x meters from one end of the rod, where k is a positive constant. How should the constant k be chosen so that the center of mass of the rod is one meter from the other end of the rod.

Let x be the distance (in meters) from one end of the rod. Then the mass of the rod is

$$M = \int_0^3 \delta(x)\,dx = \int_0^3 (1 + kx^2)\,dx = 3 + 9k$$

and the moment about $x = 0$ is

$$M_x = \int_0^3 x\delta(x)\,dx = \int_0^3 (x + kx^3)\,dx = \frac{9}{2} + \frac{81k}{4}.$$

Since the center of mass $\overline{x} = M_x/M$ is to be at $\overline{x} = 2$, one meter from the other end of the bar, we must have

$$2 = \frac{\frac{9}{2} + \frac{81k}{4}}{3 + 9k}$$

or $9/2 + 81k/4 = 6 + 18k$. Solving for k, we find $k = 2/3$.

8. (15 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**.

(a) If the power series $\sum_{n=0}^{\infty} C_n x^n$ is known to converge at 1 and to diverge at -2, then we can conclude that the power series diverges at 3.

$\boxed{\textbf{T}}$ **F**

(b) If $p(x)$ denotes a density function defined on the interval $[a, b]$, and $P(x)$ denotes an antiderivative of $p(x)$, then the function $P(x) - P(b) + 1$ is the cumulative distribution function for $p(x)$.

$\boxed{\textbf{T}}$ **F**

(c) If the pair of functions $(x(t), y(t))$ gives a parameterization of the unit circle centered at the origin, then the integral $\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is equal to 2π.

T $\boxed{\textbf{F}}$

(d) If $P_2(x)$ is the second degree Taylor polynomial that approximates a function f about $x = 3$, and if $E_2(x) = f(x) - P_2(x)$, is the error in the approximation of f by P_2, then $E_2(3) = 0$, $E_2'(3) = 0$, and $E_2''(3) = 0$.

$\boxed{\textbf{T}}$ **F**

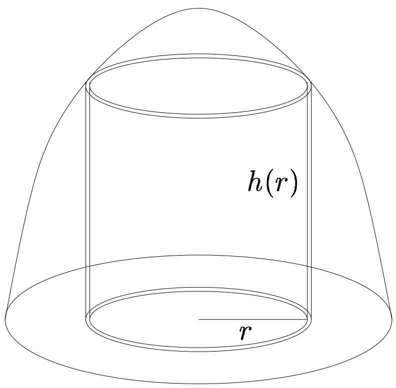
(e) If r and a are any positive numbers, then $\sum_{n=0}^{\infty} a\, r^n = \frac{a}{1-r}$.

T $\boxed{\textbf{F}}$

9. (10 points) Paleontologists have found some interesting fossils embedded in a stalagmite in a cave. To help determine the age of the fossils they want to measure the mass of the stalagmite.

At its base, the stalagmite has a radius of 50 cm and its height $h(r)$, r cm from its center is $\dfrac{r^2 - 100r + 2500}{50}$ cm. The density of the limestone from which the stalagmite is made r cm from its center is $\dfrac{50 + r}{50}$ g/cm^3. (See the figure.)

To help determine the age of the fossils, the paleontologists want to calculate the mass of the stalagmite. What is this mass?



Pictured thin clinder has walls of thickness dr.
Volume of pictured thin cylinder $= 2\pi r h(r)\, dr$.
Constant density on this cylinder $= \dfrac{50 + r}{50}$.
Mass of this cylindrical shell $= \dfrac{50 + r}{50} 2\pi r h(r)\, dr$.
Total mass of stalagmite $= \displaystyle\int_0^{50} \frac{50 + r}{50} 2\pi r h(r)\, dr = \int_0^{50} \frac{50 + r}{50} 2\pi r \frac{r^2 - 100r + 2500}{50}\, dr.$
Note that integrand $= \dfrac{2\pi}{2500}(r^4 - 50r^3 - 50^2 r^2 + 50^3 r)$.
Calculating the integral, either by computing an antiderivative and using the fundamental theorem of calculus, or but using a calculator, one finds that:

The mass of the stalagmite is $\dfrac{7 \cdot 50^3 \cdot \pi}{30} == \dfrac{87,500\pi}{3} = 91629.786$ **g.**

10. (10 points) The *electric potential* is a quantity of great importance in electrostatics. The electric potential $V(R)$ at a distance R along the axis perendicular to the center of a charged disk with radius 1 is given by

$$V(R) = C\left(\sqrt{R^2+1} - R\right)$$

where C is a constant that depends on the choice of units that are being used.

(a) Show that for large numbers R,

$$V(R) \approx \frac{C}{2R}.$$

(Hint: $\sqrt{R^2+1} = R\sqrt{1+\frac{1}{R^2}}$ and remember that R is large.)

As the hint says, $\sqrt{R^2+1} = R\sqrt{1+\frac{1}{R^2}} = R(1+\frac{1}{R^2})^{\frac{1}{2}}$.
Recall (or calculate) that for $-1 < x < 1$**, we have** $(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\ldots$
Thus for R **large (enough so that** $-1 < \frac{1}{R^2} < 1$**) we have**
$(1+\frac{1}{R^2})^{\frac{1}{2}} = 1+\frac{1}{2}\frac{1}{R^2}-\frac{1}{8}(\frac{1}{R^2})^2+\frac{1}{16}(\frac{1}{R^2})^3-\ldots = 1+\frac{1}{2}\frac{1}{R^2}-\frac{1}{8}\frac{1}{R^4}+\frac{1}{16}\frac{1}{R^6}-\ldots$
So $R(1+\frac{1}{R^2})^{\frac{1}{2}} = R+\frac{1}{2}\frac{1}{R}-\frac{1}{8}\frac{1}{R^3}+\frac{1}{16}\frac{1}{R^5}-\ldots$
Therefore $R(1+\frac{1}{R^2})^{\frac{1}{2}} - R = \frac{1}{2}\frac{1}{R}-\frac{1}{8}\frac{1}{R^3}+\frac{1}{16}\frac{1}{R^5}-\ldots$
So $C(R(1+\frac{1}{R^2})^{\frac{1}{2}} - R) = \frac{C}{2}\frac{1}{R}-\frac{C}{8}\frac{1}{R^3}+\frac{C}{16}\frac{1}{R^5}-\ldots$
And so we have that for large numbers R**, we can approximate** $C(R(1+\frac{1}{R^2})^{\frac{1}{2}}-R)$ **by** $\frac{C}{2}\frac{1}{R}$**.**

(b) Approximately how large should R be in order that the error in the approximation of $V(R)$ by $C/2R$ is less than 1% of $V(R)$?

For large R**, the error in the approximation of** $V(R)$ **by** $C/2R$ **is approximately** $\frac{-C}{8R^3}$**.**
So, we want to apporoximately solve $V(R)\frac{1}{100} = \frac{C}{8R^3}$ **for** R**.**
This is approximately the same as approximately solving $\frac{C}{2R}\frac{1}{100} = \frac{C}{8R^3}$ **for** R**.**
So we want to solve $4R^2 = 100$ **and thus**

R should approximately be greater than or equal to 5

11. (12 points) The body metabolizes a certain drug at a continuous rate of 6 percent per hour. That is, t hours after A milliliters of the drug are injected into the bloodstream, only $Ae^{-0.06t}$ milliliters of the drug are still present, the body having metabolized the remainder.

When treating patients with the drug, the preferred administration strategy is to continuously administer the drug (perhaps intravenously) at a rate of fifty milliliters (ml) per hour for ten hours. However, this is not always possible, and an alternate administration strategy is also available.

In the alternate administration strategy, an amount, S, of the drug is administered at the beginning of the ten hour period.

(a) Write a definite integral whose value is the amount of drug remaining in the bloodstream at the end of the 10 hour preferred treatment.

During a time interval of duration dt centered at time t, $50\,dt$ milliliters of the drug is entering the bloodstream.
The drug entering at this time has $10 - t$ hours until the end of the treatment
So of the drug entering the bloodstream during a time interval of duration dt centered at time t, $e^{-0.06(10-t)}50\,dt$ will remain at the end of the treatment.
"Summing", we get ...

$$\boxed{\int_0^{10} 50e^{-0.06(10-t)}\,dt}$$

(b) Calculate the amount of the drug that has been metabolized by the end of the 10 hour preferred treatment.

This is $500 -$ (the numerical value of the integral in part (a)) OR
$$\int_0^{10} 50(1 - e^{-0.06(10-t)})\,dt$$
This is equal to $500 - \dfrac{2500(1 - e^{-0.6})}{3}$.
Thus the answer is

$$\boxed{\textbf{124 milliliters}}$$

(c) Calculate the amount, S, of the drug that should be administered so that a patient treated under the alternate strategy will have metabolized the same total amount of the drug over the ten hour period as a patient treated under the ideal strategy.

Under the alternate strategy, the amount remaining in the blood after 10 hours is $Se^{-0.06\cdot 10}$ milliliters.
Thus the amount metabolized is $S - Se^{-0.06\cdot 10} = S(1 - e^{-0.06\cdot 10})$ milliliters.
So we want to solve $S(1 - e^{-0.06\cdot 10}) =$ (the answer to part (b)).
Then the answer is

$$\boxed{S = \frac{500}{(1 - e^{-.6})} - \frac{2500}{3} \approx 274.85 \text{ milliliters}}$$