MATH 116 — FINAL EXAM

Solutions

Winter 2004

NAME: ______________________________ ID NUMBER: ____________________________

INSTRUCTOR: _____________________ SECTION NO: ____________________________

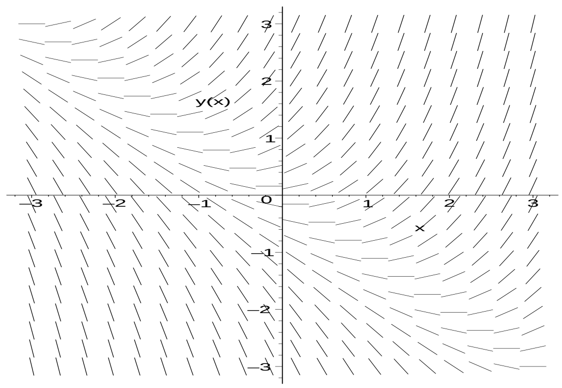
1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	4	
2	6	
3	6	
4	13	
5	10	
6	15	
7	12	
8	10	
9	10	
10	14	
TOTAL	100	

1. (4 points) Circle the differential equation whose slope field is shown in the figure.

A. $\frac{dy}{dx} = \sin x$ B. $\frac{dy}{dx} = -y$ C. $\frac{dy}{dx} = x^2 + y^2$

D. $\frac{dy}{dx} = x + y$ E. $\frac{dy}{dx} = x - 2y$ F. $\frac{dy}{dx} = \sin(x + y)$



2. (6 points) The function f is a continuous function, some of whose values are given in the following table.

x	0	1	2	3	4	5	6
$f(x)$	8	6	3	-2	0	1	2

For the function F defined by $F(x) = \int_0^x f(t)e^{-t}\,dt$, what is $F'(2)$?

$F'(2) = f(2)e^{-2} = 3e^{-2} \approx .4060058496.$

3. (6 points) Does the infinite series $\sum_{n=1}^{\infty} ne^{-n^2}$ converge or diverge? (Show your work.)

Apply the integral test. The function $f(x) = xe^{-x^2}$ is a positive, decreasing function of x for $x \geq 1$ with $f(n) = ne^{-n^2}$. Also, since $\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}$, we have by the fundamental theorem of calculus that

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty} \frac{1}{2}(e^{-1} - e^{-b^2}) = \frac{1}{2e} < \infty$$

so $\sum_{n=1}^{\infty} ne^{-n^2}$ converges by the integral test.

4. (13 points) Consider the initial value problem

$$\frac{dy}{dx} = y + e^x, \qquad y(0) = 0.$$

(a) Show that $y = xe^x$ is a solution to the initial value problem.

The value of the function $y = xe^x$ at $x = 0$ is 0, so the initial condition is satisfied. Further, by the product rule for derivatives, $\frac{dy}{dx} = e^x + xe^x$ and this is equal to $y + e^x$ so the equation is satisfied.

(b) Compute the approximation to the solution, $y(1)$, of the problem at $x = 1$ given by Euler's method with four steps (i.e. $\Delta x = \frac{1}{4}$). You must show all your work to receive credit.

With $f(x, y) = y + e^x$, and $\Delta x = .25$, we have

$$\begin{aligned}
y(.25) &\approx y(0) + f(0,0)\Delta x = 0 + (1.0)(.25) = .25 \\
y(.5) &\approx y(.25) + f(.25,.25)\Delta x = .25 + (1.5340)(.25) = .6335 \\
y(.75) &\approx y(.5) + f(.5,.6335)\Delta x = .6335 + (2.2822)(.25) = 1.2041 \\
y(1) &\approx y(.75) + f(.75,1.2041)\Delta x = 1.2041 + (3.3211)(.25) = 2.0343
\end{aligned}$$

$y(1) \approx \underline{2.0343}$

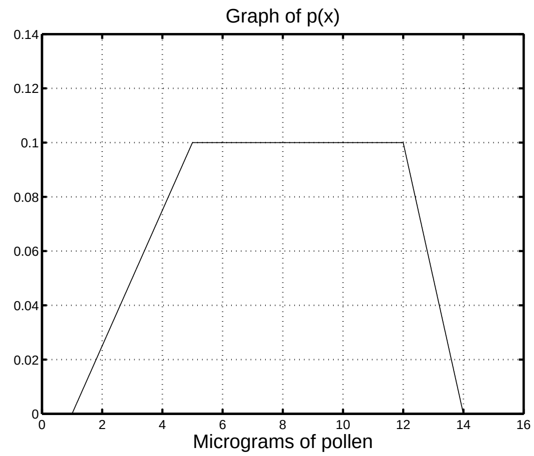
(c) What is the error when the exact solution $y(1)$ is approximated by Euler's method with 4 steps? If instead you were to use Euler's method with 16 steps, approximately what would you expect the error to be? (No explanation required. DO NOT carry out Euler's method with 16 steps.)

We learned that the error in Euler's method is approximately given by c/n where c is a constant that depends on the initial value problem and the value of x at which the solution is being approximated. Since $n = 16$ is four times larger than $n = 4$, the error should be reduced by a factor of 1/4. That is,

4-step error = *exact solution - approximate solution* $\approx e - 2.0342 \approx .6841$,
16-step error estimate $\approx$ *(4-step error)/4* $\approx .6841/4 = .1710$.

The actual error when Euler's method is carried out with 16 steps is $\approx .2003$.

5. (10 points) The number of micrograms x of pollen produced annually by plants of a certain species in a small forest has been determined (by experiment) to have a density function $p(x)$ whose graph is shown in the figure.



(a) Write a definite integral whose value is the fraction of the plants that produce less than 10 micrograms of pollen each year.

$\int_0^{10} p(x)\,dx$ *where* $p(x)$ *is the function whose graph is given.*

(b) What fraction of the plant population produces less than 10 micrograms of pollen each year?

$$\int_0^{10} p(x)\,dx = \text{ area under graph between } x = 0 \text{ and } x = 10$$
$$= .2 + .5 = .7$$

or 70% of the population.

(c) Let $P(x)$ be the cumulative distribution function for this population. In terms of the population, what is the meaning of $P(13) - P(8)$?

$P(13) - P(8)$ *is the fraction of the population that produces between 8 and 13 grams of pollen each year.*

(d) What is the median number of micrograms of pollen produced by plants in this population?

The median is the value of T *for which* $P(T) = .5$*, or the value of* T *so that the area under the graph of* p *between* $x = 0$ *and* $x = T$ *is equal to* $.5$*, which is also the area under the graph of* p *between* $x = T$ *and* $x = 14$*. This number is* $T = 8$*, since the area under the graph of* p *to the left of* T *is then* $.2 + .3$ *which the area under the graph to the right of* T *is also* $.4 + .1 = .5$*.*

6. (15 points) For each of the following statements, circle **T** if the statement is always true, and otherwise circle **F**. No explanations are required.

(a) The Taylor series for $\sin(x)$ about $x = 1$ is $(x-1) - \frac{(x-1)^3}{3!} + \frac{(x-1)^5}{5!} - \cdots$

T [**F**]

(b) If Euler's method with 10 steps is used to approximate the solution to the initial value problem $\frac{dy}{dx} = -y, \quad y(0) = 1$ at $x = 1$, then the approximation will be an overestimate for the exact solution.

T [**F**]

(c) Let f be a continuous, positive, decreasing function defined for $x \geq 1$ such that $\int_1^\infty f(x)\,dx$ converges. If $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n = \int_1^\infty f(x)\,dx$.

T [**F**]

(d) The system of differential equations,

$$\begin{aligned} \frac{1}{x}\frac{dx}{dt} &= y - 1, \\ \frac{1}{y}\frac{dy}{dt} &= x - 1, \end{aligned}$$

models the interaction of two populations involved in a predator-prey relationship.

T [**F**]

(e) The relative growth rate of the population in the *logistic model for population growth* is a linear function of the population.

[**T**] **F**

7. (12 points) A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 minutes, the egg's temperature is 38°C. Assume that the water has not warmed appreciably and that the temperature of the egg changes at a rate proportional to the difference between its temperature and that of the water.

(a) Write the differential equation and initial conditions which model the temperature of the egg.

Let $T(t)$ denote the temperature of the egg t minutes after being placed in the water. Then

$$\frac{dT}{dt} = -k(T-18), \qquad T(0) = 98$$

where k is a positive constant of proportionality.

(b) Find the solution of the initial value problem of part (a). (Show your work.)

We use the method of separation of variables to solve the equation.

$$\begin{aligned} \frac{dT}{T-18} &= -kdt \\ \ln(T-18) &= -kt + C \\ T - 18 &= e^C e^{-kt} \end{aligned}$$

Since $T = 98$ when $t = 0$, we must have $A = 80$, so $T = 18 + 80e^{-kt}$.

(c) How long does it take for the temperature of the egg to reach 20°C?

We first use the fact that $T = 38$ when $t = 5$ to evaluate the constant k. That is, $38 = 18 + 80e^{-5k}$, or $1/4 = e^{-5k}$, or $k = \frac{\ln 4}{5}$. Therefore,

$$T = 18 + 80e^{-((\ln 4)/5)t}.$$

Set $T = 20$ and solve for t. The result is $t = \frac{5 \ln 40}{\ln 4} \approx 13.3048$ minutes.

8. (10 points) The *electric potential* is a quantity of great importance in electrostatics. The electric potential $V(R)$ at a distance R along the axis perpendicular to the center of a charged disk with radius 1 is given by

$$V(R) = C\left(\sqrt{R^2+1} - R\right)$$

where C is a constant that depends on the choice of units that are being used.

(a) Show that for large numbers R,

$$V(R) \approx \frac{C}{2R}.$$

(Hint: $\sqrt{R^2+1} = R\sqrt{1+\frac{1}{R^2}}$ and remember that R is large.)

As the hint says, $\sqrt{R^2+1} = R\sqrt{1+\frac{1}{R^2}} = R(1+\frac{1}{R^2})^{\frac{1}{2}}$.
Recall (or calculate) that for $-1 < x < 1$, *we have* $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$
Thus for R *large (enough so that* $-1 < \frac{1}{R^2} < 1$*) we have*
$(1+\frac{1}{R^2})^{\frac{1}{2}} = 1 + \frac{1}{2}\frac{1}{R^2} - \frac{1}{8}(\frac{1}{R^2})^2 + \frac{1}{16}(\frac{1}{R^2})^3 - \ldots = 1 + \frac{1}{2}\frac{1}{R^2} - \frac{1}{8}\frac{1}{R^4} + \frac{1}{16}\frac{1}{R^6} - \ldots$
So $R(1+\frac{1}{R^2})^{\frac{1}{2}} = R + \frac{1}{2}\frac{1}{R} - \frac{1}{8}\frac{1}{R^3} + \frac{1}{16}\frac{1}{R^5} - \ldots$
Therefore $R(1+\frac{1}{R^2})^{\frac{1}{2}} - R = \frac{1}{2}\frac{1}{R} - \frac{1}{8}\frac{1}{R^3} + \frac{1}{16}\frac{1}{R^5} - \ldots$
So $C(R(1+\frac{1}{R^2})^{\frac{1}{2}} - R) = \frac{C}{2}\frac{1}{R} - \frac{C}{8}\frac{1}{R^3} + \frac{C}{16}\frac{1}{R^5} - \ldots$
And so we have that for large numbers R, *we can approximate* $C(R(1+\frac{1}{R^2})^{\frac{1}{2}} - R)$ *by* $\frac{C}{2}\frac{1}{R}$.

(b) Approximately how large should R be in order that the error in the approximation of $V(R)$ by $C/2R$ is less than 4% of $V(R)$?

For large R, *the error in the approximation of* $V(R)$ *by* $C/2R$ *is approximately* $\frac{-C}{8R^3}$.
So, we want to solve $\frac{C}{8R^3} < .04V(R)$ *for* R.
This is approximately the same as solving $\frac{C}{8R^3} < .04\frac{C}{2R}$ *for* R.
That is, $1/4 < .04R^2$ *or* $R > 1/.4 = 2.5$. *Thus*

R should approximately be greater than 2.5

9. (10 points). The watering tank for livestock on a farm has depth of 30 inches (and a constant cross-sectional area). The tank is connected to a water source and controller so that when the depth of water in the tank is less than 25 inches, water flows into the tank at a constant rate that increases the depth of water by 4 inches per hour. When the depth of water is greater than 25 inches, no water flows into the tank.

At 6 am one morning, when the depth of the water in the tank is 25 inches, it springs a leak and the water leaks from the tank at a rate proportional to the square root of the depth of water. Let c be the proportionality constant.

(a) Write the differential equation for the depth $h(t)$ of the water in the tank at time t, where t is the time in hours after 6 am. Assume that when there is 25 inches of water in the tank, the proportionality constant c is large enough that the water leaks out faster than the four inch per hour rate at which the source adds water. (Do not attempt to solve the differential equation.)

The rate of change of $h(t)$ *is* $4 - c\sqrt{h}$ *inches per hour, so the equation is*

$$\frac{dh}{dt} = 4 - c\sqrt{h}.$$

The intitial condition would be $h(0) = 25$ *(inches).*

(b) Unfortunately, the farmer falls ill and no one checks on the tank to discover the leak for a very long time. When the leak is finally discovered, it is also found that the depth of water in the tank is 9 inches. Based on this information, estimate the constant of proportionality, c, that determines the rate that water leaks from the tank? Explain how you found your answer.

The equilibrium solution of the equation is the solution to $4 - c\sqrt{h} = 0$ *or* $h(t) \equiv (4/c)^2$. *Further, for* $(4/c)^2 < h < 25$, *the solution* $h(t)$ *has* $dh/dt = 4 - c\sqrt{h} < 0$, *so* $h(t)$ *decreases to the equilibrium solution* $(4/c)^2$. *That is, after a long time, we have* $h(t) \approx (4/c)^2$. *From the information given in part (b), we have* $(4/c)^2 = 9$ *inches, or* $c = 4/3$.

10. (14 points) A cylindrical tank is has a circular cross section of radius 2 meters and a length of 4 meters. It is to be filled with a compressible liquid whose density varies with its height and is equal to $\rho(h) = 60\sqrt{1+h}$ kg/m^3 at h meters below the surface of the liquid.

(a) Suppose the tank is standing on one of its circular ends (figure 1) and is filled with the liquid. What is the approximate mass in a thin horizontal slice of thickness Δh that is h meters below the top of the tank?

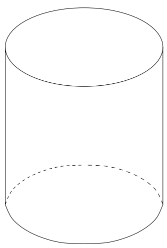


Figure 1

Slice the liquid in the tank into smaller cylinders by planes perpendicular to the axis of the cylinder. Each such thin slice has a circular cross section of radius 2 m and a thickness of Δh, where h measures the distance from the top of the cylinder. The volume of the slice is equal to

$$\text{area of base} \times \text{height} = (\pi 2^2)\Delta h = 4\pi\Delta h.$$

The density of the liquid is almost constant throughout the thin slice h m $(0 \le h \le 4)$ below the top of the tank, with a value of $60\sqrt{1+h}$ kg/m^3. Hence,

$$\text{amount of liquid in slice} = \text{density} \times \text{volume} \sim 60\sqrt{1+h} \times 4\pi\Delta h = 240\pi\sqrt{1+h}\Delta h \ kg.$$

(b) Write a definite integral whose value is equal to the total mass of the liquid in the tank.

The total amount of liquid in the tank is obtained by adding up the amount of liquid is each of the slices, so is given by the Riemann sum

$$\sum \text{amount of liquid in slice} \sim \sum 240\pi\sqrt{1+h}\Delta h$$

which, as we cut the liquid into thiner and thiner slices, tends to the integral

$$\text{amount of liquid in tank} = \int_0^4 240\pi\sqrt{1+h}\, dh \ kg.$$

(c) Evaluate your integral from part (b) to find the total amount of liquid in the tank. Show your work, or explain how you obtained your answer.

We can evaluate the integral using the fundamental theorem of calculus.

$$\int_0^4 240\pi\sqrt{1+h}\, dh = 160\pi(1+h)^{3/2}\Big|_0^4 = 160\pi(5\sqrt{5}-1) \approx 5117.19696 \text{ kg}.$$

(d) Suppose that instead the tank is lying on a side (figure 2) and again filled with the liquid. What then is the approximate mass of the liquid in a thin horizontal slice of thickness Δh that is h meters below the top of the tank?

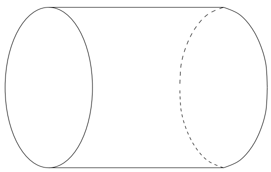


Figure 2

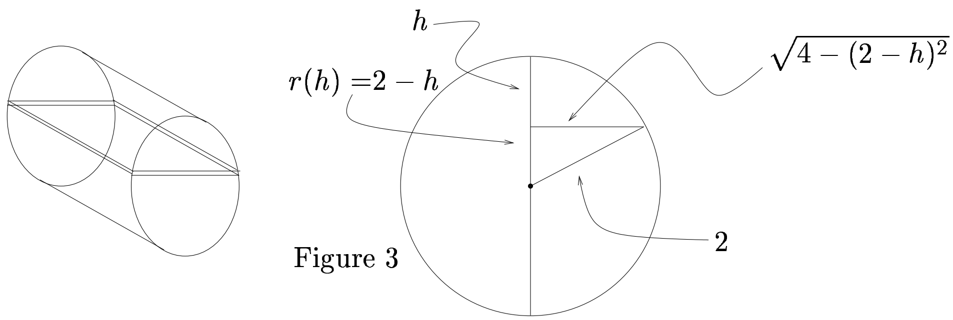


Figure 3

This time the slices have a rectangular cross section, with one side of length 4 and the other of length $2 \times \sqrt{4 - r^2}$. (See figure 3), where $r = r(h) = 2 - h$ if $0 < h < 2$ and $r(h) = h - 2$ if $2 < h < 4$. That is, $r(h) = |2 - h|$. The volume of the slice is equal to

$$\text{area of base} \times \text{height} = 4 \times (2\sqrt{2^2 - |h-2|^2}) \times \Delta h = 8\sqrt{4 - |h-2|^2}\Delta h.$$

The density of the liquid is almost constant throughout the thin slice h m $(0 \le h \le 4)$ below the top of the tank, with a value of $60\sqrt{1+h}$ kg/m^3. Hence,

$$\begin{aligned}\text{amount of liquid in slice} &= \text{density} \times \text{volume} \sim 60\sqrt{1+h} \times 8\sqrt{4-(2-h)^2}\delta h \\ &= 480\sqrt{(1+h)(4h-h^2)}\ kg.\end{aligned}$$

(e) Is the amount of liquid the same as in part (c)? If so, explain why. If not, find the amount of liquid in the tank.

There does not appear to be any obvious reason the amount of liquid should be the same in the two circumstances. So, let us compute once again the volume of the liquid in the tank. Summing up the volumes of the slices and passing to the limit as $\Delta h \to 0$ shows that the volume of liquid (in kg) is equal to

$$\int_0^4 480\sqrt{h(1+h)(4-h)}\, dh \approx 5145.223877 \text{ kg}.$$

which is about 28 kg more than quantity computed in part (a). The tank holds more lying on its side than on its end.

What would happen if you put the tank on its side, filled it up, and then stood it up on end?