Math 116 — Exam 1

February 9th, 2009

Name: ____

Instructor: ______________________________ Section: ______________

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1. **Do not open this exam until you are told to do so**.
2. This exam has 9 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. One of the skills being tested on this exam is your ability to know when you must make an estimation vs. an exact calculation. Some problems will require estimations. Others will have exact answers. Methods that yield exact answers are will be given more credit than an estimation, if an exact answer is available.
7. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, *you must show work for any calculation we have learned how to do in this course*.
8. You are allowed two sides of a 3” x 5” note card.
9. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
10. **Turn off all cell phones and pagers, and remove all headphones**.

Problem	Points	Score
1	85	
2	30	
3	35	
4	40	
5	40	
6	20	
Total	**250**	

1. (85 points) **Modeling the amount of water in a container.** Consider 3 containers, in which water flows into or out of each container at a different rate. Your job is to determine how much water is in each container at the end of 75 seconds.

   a. If $r(t)$ describes the flow of water into a container with units of milliliters per second (ml/sec), and t is measured in seconds, write a sentence or two explaining what $\int_a^b r(t)dt$ means in this context.

   This definite integral represents the net change in the amount of water (in ml) in the container between $t = a$ seconds and $t = b$ seconds

   b. ***Container 1*****:** The initial amount of water in container 1 is 150 milliliters (ml). Water flows into container 1 at a rate $r_1(t)$ ml/sec described by the following data.

Time (sec)	0	25	50	75
$r_1(t)$ (ml/sec)	23	21	6	2

   What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

   LHS: 150+23(25)+21(25)+6(25)=150+1250=1400 ml. This is an overestimate if we assume that the data is decreasing always.

   RHS: 150+21(25)+6(25)+2(25)=150+725=875 ml. This is an underestimate if we assume that the data is always increasing.

   TRAP=(RHS+LSH)/2=987.5. If students assume data is linear, then trap is exact. Otherwise, we need more info to know if this estimation is over/under.

c. ***Container 2:*** The initial amount of water in container 2 is 150 milliliters (ml). Water flows into container 2 at a rate $r_2(t)$ ml/sec. *An anti-derivative* of $r_2(t)$ is $R_2(t) = \frac{100t}{35}\sin\left(\frac{t}{18}+3\right)$. What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

150+ $R_2(75) - R_2(0) \approx$ 150+165.63=315.63. This is an exact method. But round-off error may occur, depending on how student writes down answer.

d. ***Container 3:*** The initial amount of water in container 3 is 150 milliliters (ml). Water flows into container 3 at a rate $r_3(t) = \frac{50}{t^2+5t+6} + 10\sin\left(\frac{2\pi}{75}t\right)$ ml/sec. What is the volume of water at the end of 75 seconds? Describe the method you use, and the accuracy of your method (i.e. exact, over/underestimate, etc.). If you've made assumptions that affect your answer, you should also explain those as well.

Find the anti-derivative of $r_3(t)$ and use FTC. This requires partial fractions and a straightforward w-substitution.

$$\int_0^{75} \frac{50}{t^2+5t+6} + 10\sin\left(\frac{2\pi}{75}t\right) dt = \int_0^{75} \frac{50}{t+2} - \frac{50}{t+3} + 10\sin\left(\frac{2\pi}{75}t\right) dt$$

$$= 50\ln|t+2| - 50\ln|t+3| - \frac{750}{2\pi}\cos\left(\frac{2\pi}{75}t\right)\Bigg|_0^{75}$$

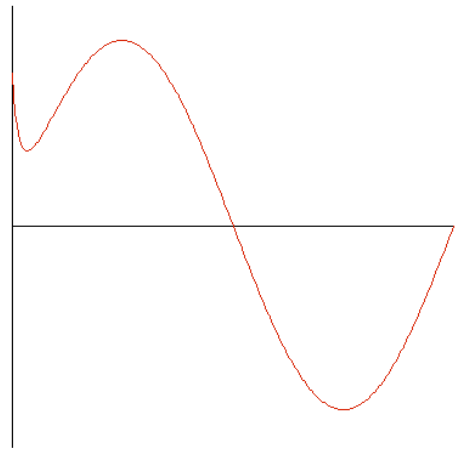
$$= \left(50\ln\left(\frac{77}{78}\right) - \frac{750}{2\pi}\cos(2\pi)\right) - \left(50\ln\left(\frac{2}{3}\right) - \frac{750}{2\pi}\cos(2\pi)\right)$$

$$= 50\ln\left(\frac{77}{52}\right) \approx 19.62$$

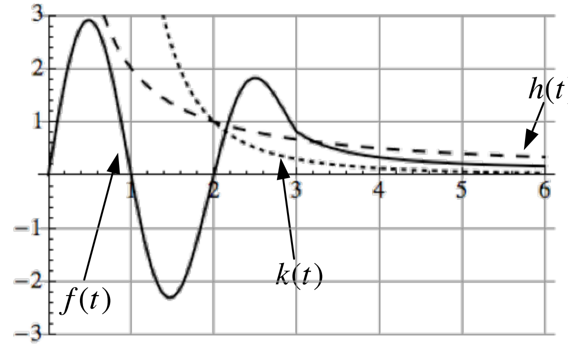
Add this to 150 to get 169.62. This method is exact.

e. Considering only the first 75 seconds, does container 3 have its maximum amount of water at 75 seconds? Justify your response.

No, $r_3(t)$ becomes negative prior to 75 seconds. This means water is leaving the container when the rate function is negative. I can graph $r_3(t)$ and see that $r_3(t)$ is negative from about 38 seconds to 75 seconds. This means that the container is losing water and cannot have its maximum at 75 seconds.



2. (30 points) The graphs of $f(t)$, $h(t)$, and $k(t)$ are shown below. You may assume that as $t \to \infty$, the graphs of f, h, and k continue in a fashion similar to the trend observed in the graph on the right. We define $g(x) = \int_1^{x^2} f(t)\,dt$.



a. What's $g'(2)$?

By FTC, $g'(x) = 2xf(x^2)$. Thus

$g'(2) = 4f(4) \approx 4\left(\frac{1}{3}\right)$.

b. What, if anything, could you say about $\lim_{x\to\infty} g(x) = \int_1^{\infty} f(t)dt$ if you knew that $h(t) < \frac{1}{t\sqrt{t}}$ for $t \geq 6$? Explain your answer.

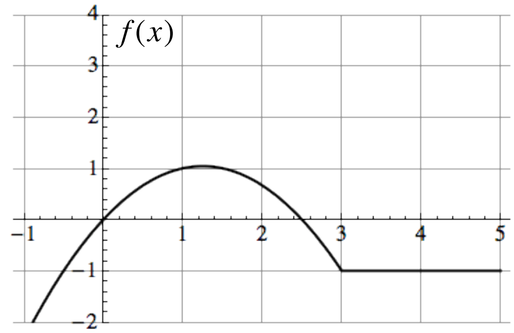
<u>It converges</u>. The graph indicates that $f(t) < h(t)$ for $t > 6$. Thus $\int_6^{\infty} f(t)dt < \int_6^{\infty} h(t)dt = \int_6^{\infty} \frac{1}{t^{\frac{3}{2}}}\,dt$.

The last integral converges since $p = \frac{3}{2}$ and by the comparison test, $\int_6^{\infty} f(t)dt$ must converge as well. Since $\int_1^{\infty} f(t)dt = \int_1^{6} f(t)dt + \int_6^{\infty} f(t)\,dt$, we're adding only a finite amount of area and thus $\int_1^{\infty} f(t)dt$ converges.

c. What, if anything, could you say about $\lim_{x\to\infty} g(x) = \int_1^{\infty} f(t)dt$ if you were to instead assume that $\int_{100}^{\infty} k(t)dt = 16$? Explain your answer.

<u>Inconclusive.</u> The graph indicates that $f(t) > k(t)$ and thus $\int_{100}^{\infty} f(t)dt > \int_{100}^{\infty} k(t)dt$. The fact that $\int_{100}^{\infty} k(t)dt = 16$ means that the integral converges. But this is not enough to determine whether or not $\int_{100}^{\infty} f(t)dt$ converges. And this means that we don't have enough info to determine whether or not $\int_1^{\infty} f(t)dt$ converges

3. (35 points) The graph of $f(x)$ on $[0,4]$ is shown below. You may want to refer to it to answer the following questions.



a. What is $\int_0^4 x f'(x)\, dx$?

Using integration by parts,
$$\int_0^4 f(x)dx \approx xf(x)\Big|_0^4 - \int_0^4 f(x)dx$$

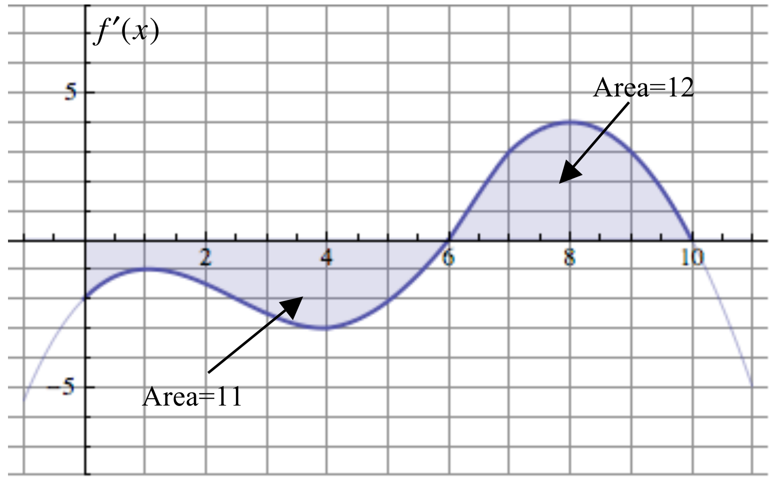
Using geometry, I estimate that $\int_0^4 f(x)dx \approx \frac{1}{2}(2.5)(1) - \frac{1}{2}(0.5)(1) - 1(1) = 0$

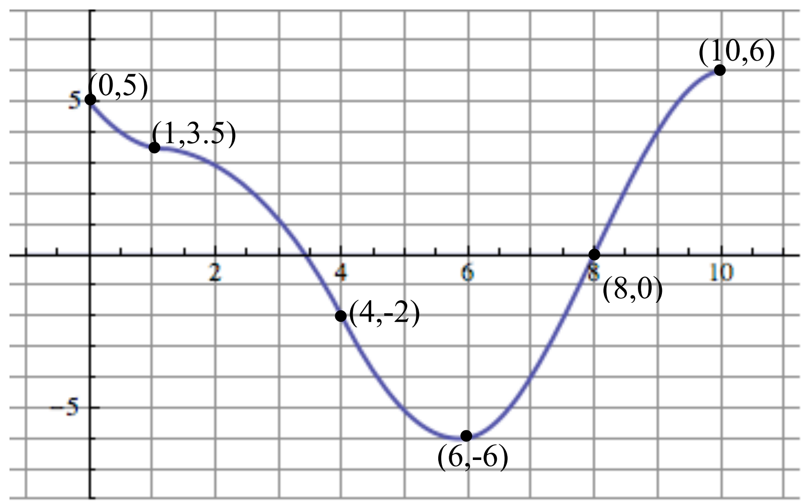
So $\int_0^4 xf'(x)dx \approx 4f(4) - 0f(0) - (0) = 4(-1) - 0 = -4$

b. What is $\int_0^2 x f'(x^2)\, dx$?

Using w-substitution, $\int_0^2 x f'(x^2)dx = \frac{1}{2}\int_0^4 f'(w)dw = \frac{1}{2}[f(4) - f(0)]$
$= \frac{1}{2}[-1-0] = -\frac{1}{2}$

4. (40 points) The graph of $f'(x)$ is shown in the graph below. Given the fact that $f(0)=5$, sketch a rough graph of $f(x)$ on the blank axes provided for the domain $[0,10]$. You should indicate all critical points, inflection points, and function values (if applicable).





Students should note (graphically or otherwise) that the function is decreasing on [0,6] and increasing on [6,10]. There's a local min at *x*=6. The function is concave up on $[0,1]\cup[4,8]$ and concave down on $[1,4]\cup[8,10]$. In addition to $f(0)=5$, we can use FTOC to find other function values. For example, $-11=\int_0^6 f'(x)dx = f(6)-f(0)$ and $f(0)=5$ to get $f(6)=-6$. Similarly $f(10)=6$. Other relevant points are inflection points at (1,3.5), (4,-2), and (8,0) (found by estimating area under the graph). Critical points exist at (6,-6) and (10,6) since $f'=0$.

5. **True/False/Explain** (40 points) For each of the following determine whether the statements are true or false. To receive credit you must justify your decision with a relevant sentence or two, calculation or picture explaining your thoughts.

   a. Suppose that a function h and its derivative h' are continuous. If $h'(x) < 0$ for all $a \le x \le b$ then every left-hand sum estimate of $\int_a^b h(x)\,dx$ will be an overestimate.

   TRUE $h'(x) < 0$ means that *h* is strictly decreasing. Thus, every left-had sum will be an overestimate.

   b. If $f(x)$ is continuous on $[-5,5]$, then $\int_0^2 |f(x)|\,dx \le \int_0^3 |f(x)|\,dx$

   TRUE. $\int_0^3 |f(x)|\,dx = \int_0^2 |f(x)|\,dx + \int_2^3 |f(x)|\,dx \ge \int_0^2 |f(x)|\,dx$. The last inequality is true since $\int_2^3 |f(x)|\,dx$ must be zero or greater due to the fact $|f(x)|$ is zero or greater.

   c. If $f(x)$ is a positive, continuous function for $x \ge 0$, and if $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ converges.

   FALSE. If $f(x) = \frac{1}{x}$, $\lim_{x\to\infty} f(x) = 0$ but $\int_1^\infty f(x)dx$ diverges (*p*=1).

   d. If $F(x)$ and $G(x)$ are anti-derivatives of a function $f(x)$ that is continuous on $(-\infty,\infty)$, and if $F(5) > G(5)$, then $F(10) > G(10)$.

   TRUE. $F(x) = G(x) + C$ since *F* and *G* are anti-derivatives of the same function. Since $F(5) > G(5)$, the constant *C* must be positive. That means that since $F(10) = G(10) + C$, $F(10) > G(10)$

6. (20 points) The quantity $\int_1^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ roughly models the resistance that football-shaped plankton encounter when falling through water. Note that $a=1$, $b=2$, and $c=3$ are constants that describe the dimensions of the plankton.

   Find a value of *M* for which $\int_1^{M} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ differs from the original model of resistance by at most 0.001. *Hint*: make use of the integral $\int_M^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$ and the comparison test.

We want $\int_1^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} - \int_1^{M} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}}$

$$= \int_M^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < 0.001$$

We observe that $\int_M^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < \int_M^{\infty} \frac{dx}{\sqrt{(x)(x)(x)}} = \int_M^{\infty} \frac{1}{x^{\frac{3}{2}}}dx$ since a^2, b^2, and c^2 are all positive constants. So if $\int_M^{\infty} \frac{1}{x^{\frac{3}{2}}}dx < 0.001$ then $\int_M^{\infty} \frac{dx}{\sqrt{(a^2+x)(b^2+x)(c^2+x)}} < 0.001$.

Convert to a proper integral and solve:

$$\lim_{b\to\infty}\int_M^{b} x^{-\frac{3}{2}}dx = \lim_{b\to\infty} -2x^{-\frac{1}{2}}\Big|_M^b = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}}+\frac{2}{\sqrt{M}}\right) = \frac{2}{\sqrt{M}} < 0.001.$$

Algebra yields $M > \left(\frac{2}{0.001}\right)^2 = 4{,}000{,}000$