Math 116 — Exam 2

March 24th, 2009

Name: ______________________________

Instructor: ______________________________ Section: ______________

1. **Do not open this exam until you are told to do so**.
2. This exam has 9 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. One of the skills being tested on this exam is your ability to know when you must make an estimation vs. an exact calculation. Some problems will require estimations. Others will have exact answers. Methods that yield exact answers are will be given more credit than an estimation, if an exact answer is available.
7. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, *you must show work for any calculation we have learned how to do in this course.*
8. You are allowed two sides of a 3" x 5" note card.
9. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
10. **Turn off all cell phones and pagers, and remove all headphones**.

Problem	Points	Score
1	30	
2	60	
3	60	
4	50	
5	52	
6	48	
Total	**300**	

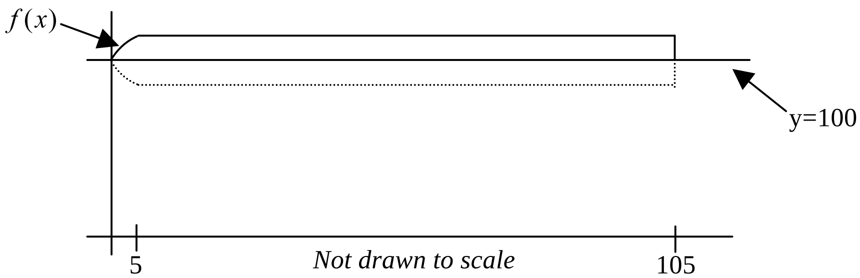
1. (30 points) A proposed Math 116 stimulus package consists of the government giving \$100 billion to individuals in the U.S. via a tax cut. We suppose that all those receiving a tax cut spend 80% of the money they get and save 20% of it. The 80% that they spend is then going to other people (shop owners, employees, etc.). These other people then spend 80% of what they receive and save 20%. This continues on indefinitely. Calculate the total additional spending created by this \$100 billion tax cut.

   Total spending is $100(0.8)+100(0.8)^2+100(0.8)^3+100(0.8)^4+\cdots$.
   To compute total, We will examine the sum or the first *n* terms.
   $S_n = 100(0.8)+100(0.8)^2+100(0.8)^3+100(0.8)^4+\cdots+100(0.8)^n$
   So $(0.8)S_n = 100(0.8)^2+100(0.8)^3+100(0.8)^4+\cdots+100(0.8)^{n+1}$ and
   $S_n-(0.8)S_n = 100(0.8)-100(0.8)^{n+1}$ which, after some algebra, yields
   $S_n = \dfrac{100(0.8)-100(0.8)^{n+1}}{0.2}$. Take the limit as *n* goes to infinity, and $S_\infty = \dfrac{100(0.8)}{0.2} = 400$ billion dollars.

2. (60 points) Several different forces act on a submerged submarine and cause it to rise and/or fall. In this problem, we will use a simplified model of a submarine to explore some of these forces. To construct our model, we revolve the graph of

$$f(x)=\begin{cases} 100+\sqrt{5}e^{0.5}\sqrt{x}e^{-0.1x} & 0\le x\le 5 \\ 105 & 5<x\le 105 \end{cases}$$

around $y=100$ (see picture below). Note that all units in the horizontal and vertical directions are measured in meters. We will also assume that ocean water has density of $1025\,{}^{kg}\!/_{m^3}$ and that the density of material inside the submarine is a constant represented by the symbol δ. Additionally, the acceleration due to gravity is $9.8\,{}^{m}\!/_{sec^2}$.



a. The *force due to buoyancy* is an upward force equal to the weight of the water displaced by the volume of the submarine. Find the volume of water displaced by the sub (i.e. the volume of the submarine) and the resulting force due to the buoyancy.

On $x\in[0,5]$,

$V_{slice}=\pi r^2\Delta x=\pi\left(f(x)-100\right)^2\Delta x=\pi 5e^{1.0}xe^{-0.2x}\Delta x$.

So total volume is $\lim\limits_{\Delta x\to 0}\sum \pi 5e^{1}xe^{-0.2x}\Delta x=5\pi e^{1}\int_0^5 xe^{-0.2x}\,dx$. Use integration by parts with $u=x$ and $v'=e^{-0.2x}$ to get

$$5\pi e\int_0^5 xe^{-0.2x}\,dx=5\pi e\left[x\frac{e^{-0.2x}}{-0.2}\Bigg|_0^5-\int_0^5\frac{e^{-0.2x}}{-0.2}\,dx\right]$$

$$=5\pi e\left[x\frac{e^{-0.2x}}{-0.2}-\frac{e^{-0.2x}}{0.04}\right]_0^5=5\pi e\left[5\frac{e^{-1}}{-0.2}-\frac{e^{-1}}{0.04}-0+\frac{e^{0}}{0.04}\right]$$

$$=-250\pi+125\pi e=282.0686m^3$$

On $x\in(5,105]$, we've got a cylinder with radius 5. So volume is $2500\pi m^3$

Total volume is $2250\pi+125\pi e=8136.05m^3$

Convert this to a force by multiplying by density and acceleration due to gravity to get

$1025\cdot 9.8\cdot\left(2250\pi+125\pi e\right)=81,726,624.745N$

2. (b) Find the center of mass of the nose of the submarine (i.e. the shape of the first 5 meters of our model). Note *it is only necessary to set up, but not calculate*, (an) integral(s).

$$\bar{x} = \frac{\int_0^5 x\delta A_x(x)\,dx}{\text{mass}} = \frac{\int_0^5 x\delta\pi\left(\sqrt{5}e^{0.5}\sqrt{x}e^{-0.1x}\right)^2\,dx}{\text{mass}} = \frac{5e\pi\delta\int_0^5 x^2e^{-0.2x}\,dx}{\text{mass}}$$

where

$$\text{mass} = 1025(2250\pi + 125\pi e) = 8339451.5\,\text{kg}.$$

By symmetry, $\bar{y} = \bar{z} = 0$. In this case, I'm assuming the z direction is perpendicular to the usual x and y axes.

3. (60 points) The following questions refer to the submarine described in problem #2.

(a) The buoyancy properties of the empty submarine described in problem 2 cause the submarine to begin moving upward through the ocean water. This motion, in conjunction with the ocean water, creates a damping force that begins to slow the submarine. Assume that the damping force is proportional to the square of the velocity of the submarine, and that when the velocity is 5 m/s the force is 100 N. For our model submarine, the velocity at t seconds can be described by

$$v(t) = \left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)^{\frac{1}{3}} \quad \text{meters per second.}$$

Find the amount of work the damping force does on the submarine over the first 30 seconds of motion.

From the problem statement, the damping force is kv^2 where k is the proportionality constant. Since the force is 100 N when the velocity is 5 m/s, we solve and find that

$$k = \frac{100\,\text{N}}{(5\,\text{m/s})^2} = 4\,\text{kg/m}.$$

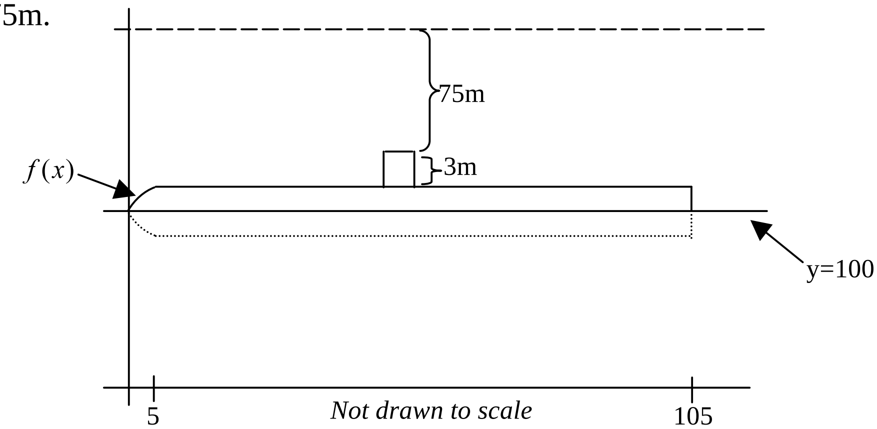
The distance travelled from time t to time $t + \Delta t$ is approximately $v(t)\Delta t$. Thus the work done over that slice of time is

$$\text{Force}\cdot\text{Distance} = (kv^2)(v\Delta t) = kv^3\Delta t = 4\left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)\Delta t$$

which means the total work is

$$\begin{aligned}\int_0^{30} 4\left(25 - 25\sin\left(\frac{\pi t}{60}\right)\right)dt &= 100\int_0^{30}\left(1 - \sin\left(\frac{\pi t}{60}\right)\right)dt = 100\left[t + \frac{60}{\pi}\cos\left(\frac{\pi t}{60}\right)\right]_0^{30}\\ &= 100\left[\left(30 + \frac{60}{\pi}\cos\frac{\pi}{2}\right) - \left(0 + \frac{60}{\pi}\cos 0\right)\right]\\ &= 100(30 - 60/\pi) \approx \boxed{1090.14\,\text{Joules}}.\end{aligned}$$

b. The *sail* of a submarine is a tower that houses the command and communications center, periscope(s), radar and antennae. We will additionally assume our model submarine has a sail that is a circular cylinder with radius of 2m and a height of 3m. Determine the total force on the sail (i.e. top and side) due to water pressure when the top of the sail is at a depth of 75m.



Pressure= mass density x acceleration due to gravity x depth.
Force=pressure x area.
So Force= mass density x acceleration due to gravity x depth x area.

Force on the side of the sail.
We will slice the sail vertically, with h = 0 located at the bottom of the sail. We also assume that up is in the positive direction.
This yields circular slices. So $Force_{slice} = 1025 \cdot 9.8 \cdot (78 - h) \cdot (2\pi \cdot 2\Delta h)$, where $(2\pi \cdot 2\Delta h)$ is the area of a circular strip of radius 2 and height Δh. The total force is found by adding up the force on each slice and taking the limit as $\Delta h \to 0$. Symbolically, this is

$$\lim_{\Delta h \to 0} \sum 1025 \cdot 9.8 \cdot (78 - h) \cdot (2\pi \cdot 2\Delta h) = 1025 \cdot 9.8 \cdot 4\pi \int_0^3 (78 - h)\,dh$$

Finding anti-derivatives yields $1025 \cdot 9.8 \cdot 4\pi \left[78h - \frac{h^2}{2} \right]_0^3$ =40,180

$40{,}180\pi \cdot 229.5 = 9{,}221{,}310\pi = 28{,}969{,}599.752$ Units are Newtons.

Force on the top. Depth is 75m and area is $\pi(2)^2$. So force on top is
$1025 \cdot 9.8 \cdot 75 \cdot 4\pi = 3{,}013{,}500\pi = 9{,}467{,}189.462N$.

The total is $12{,}234{,}810\pi = 38{,}436{,}789.21N$

4. (50 points) The *Erlang k-distribution* is a probability distribution often used in mathematical modeling when events happen at a roughly (but not exactly) constant rate. It is a good model for the wait times at a telephone switchboard when calls come in on average every λ seconds. In this case, the wait time for the next *k* telephone calls has a probability density function that is the Erlang k-distribution

$$f_{k,\lambda}(x) = \begin{cases} \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

a. For $x \geq 0$, the Erlang k- distribution has the non-obvious cumulative distribution function,

$$C_{k,\lambda}(x) = 1 - \sum_{n=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^n}{n!}.$$

Using an appropriate test, show that the sum in the cumulative distribution function converges as $k \to \infty$, thus verifying that $C_{k,\lambda}(x)$ is finite. You may assume $x = 1$ and $\lambda = 3$. Note: saying "the distribution function must be finite, therefore it converges" will not be given credit.

As $k \to \infty$, the sum in the cumulative distribution becomes $\displaystyle\sum_{n=0}^{\infty} \frac{e^{-3} 3^n}{n!}$

Using the ratio test, we get $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^{-3}3^{n+1}}{(n+1)!}\cdot\frac{n!}{e^{-3}3^n}\right| = \lim_{n\to\infty}\left|\frac{3}{n+1}\cdot\frac{1}{1}\right| = 0$ since the numerator is constant and the denominator grows to infinity.

Since $L = 0 < 1$, the ratio test allows us to conclude that the original sum *converges*

b. A call arrives at the switchboard at 2:38:06pm. Assuming $k = 1$ and $\lambda = 3$ seconds, find the probability that the next phone call comes in between 2:38:08pm and 2:38:09pm.

Using the c.d.f, we simply subtract

$$C_{1,3}(3) - C_{1,3}(2) = -\frac{e^{-3(3)}(3\cdot 3)^0}{0!} + \frac{e^{-3(2)}(3\cdot 2)^0}{0!} = \frac{-1}{e^9} + \frac{1}{e^6} = .002355$$ to get 0.2355%

Using the p.d.f, we compute

$$\int_2^3 f_{1,3}(x)dx = \int_2^3 \frac{3^1 x^0 e^{-3x}}{0!}dx = 3\int_2^3 e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_2^3 = -e^{-9} + e^{-6} = 0.002355$$ which is 0.2355%

5. (52 points) Rigorously determine whether or not the following converge or diverge. You should make clear to the grader any thoughts and/or processes that you use.

a. $\displaystyle\sum_{n=6}^{\infty}\frac{\ln(n)+3}{n-4}$

Note that $\dfrac{\ln(n)+3}{n-4}>\dfrac{\ln(n)}{n-4}>\dfrac{1}{n-4}>\dfrac{1}{n}$. This implies that $\displaystyle\sum_{n=6}^{\infty}\frac{\ln(n)+3}{n-4}>\sum_{n=6}^{\infty}\frac{1}{n}$. We know by the *p*-test that $\displaystyle\sum_{n=6}^{\infty}\frac{1}{n}$ diverges (p = 1). Thus by the comparison test, since $\displaystyle\sum_{n=6}^{\infty}\frac{\ln(n)+3}{n-4}$ is larger than $\displaystyle\sum_{n=6}^{\infty}\frac{1}{n}$, $\displaystyle\sum_{n=6}^{\infty}\frac{\ln(n)+3}{n-4}$ must diverge as well.

b. $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sin(n)+1}{e^n-n-1}$

We use the limit comparison test with $b_n=\dfrac{n}{e^n}$. Now

$\displaystyle\lim_{n\to\infty}\frac{\dfrac{n+\sin(n)+1}{e^n-n-1}}{\dfrac{n}{e^n}}=\lim_{n\to\infty}\frac{e^n}{e^n-n-1}\cdot\frac{(n+\sin(n)+1)}{n}$ But $\dfrac{e^n}{e^n-n-1}$ approaches 1 as n gets large since exponential functions dominate polynomial functions. And since $-1\le\sin(n)\le 1$, $\dfrac{(n+\sin(n)+1)}{n}$ approaches 1 as n grows large. Thus $\displaystyle\lim_{n\to\infty}\frac{e^n}{e^n-n-1}\cdot\frac{(n+\sin(n)+1)}{n}=1\cdot 1=1$. Since this limit is finite and non-zero, we know that $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sin(n)+1}{e^n-n-1}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ both converge or both diverge. Apply ratio test .

$\displaystyle\lim_{n\to\infty}\left|\frac{n+1}{e^{n+1}}\cdot\frac{e^n}{n}\right|=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{1}{e}\right|=\frac{1}{e}<1$. Since the limit is less than 1, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ converges by the ratio test and thus by the limit comparison test $\displaystyle\sum_{n=1}^{\infty}\frac{n+\sin(n)+1}{e^n-n-1}$ converges.

6. (48 points) Given $\sum_{n=1}^{\infty} a_n = 0.72$, $b_n = n^2$, $c_n = (n+1)^3$ determine whether or not the following statements are is True or False. To receive full credit, you must justify your decision with a calculation, a sentence or two, or a relevant picture that illustrates your thinking.

   a. $\lim_{n\to\infty} a_n = 0.72$. FALSE. Individual terms of a sequence must approach zero in order for the series to converge.

   b. $a_{n+1} < a_n$ for all *n*. FALSE. The sequence $a_1 = 0$, $a_2 = .72$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$, etc. has sum $\sum_{n=1}^{\infty} a_n = 0.72$. But $a_{n+1} < a_n$ does not hold for all *n*.

   c. $\lim_{n\to\infty} s_n = 0.72$ where $s_n = a_1 + a_2 + ... + a_n$ TRUE. This is the definition of convergence of a series.

   d. $\lim_{n\to\infty} \frac{b_n}{c_n}$ converges. TRUE. $\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^3} = 0$ since the denominator has a polynomial with larger degree than the polynomial in the numerator.

   e. $\sum_{n=1}^{\infty} \frac{b_n}{c_n}$ converges. FALSE. $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^3} > \sum_{n=1}^{\infty} \frac{n^2}{(n)^3} = \sum_{n=1}^{\infty} \frac{1}{n}$. But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p = 1) and by comparison test so must $\sum_{n=1}^{\infty} \frac{b_n}{c_n}$.

   f. $\sum_{n=1}^{\infty} (-1)^n \frac{b_n}{c_n}$ converges. TRUE. This is an alternating series. $\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^3} = 0$ and $\frac{n^2}{(n+1)^3}$ decreases to zero as *n* increases. So by alternating series test, the series converges.