

# Math 116 — Final Exam

April 23, 2010

Name: \_\_\_\_\_

Instructor: \_\_\_\_\_ Section: \_\_\_\_\_

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a  $3'' \times 5''$  note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

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Problem	Points	Score
1	9	
2	18	
3	12	
4	8	
5	8	
6	10	
7	11	
8	12	
9	12	
Total	100	

You may find the following expressions useful. And you may not. But you may use them if they prove useful.

**“Known” Taylor series (all around  $x = 0$ ):**

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$$

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**“Known” equations from geometry:**

Volume of a sphere:  $V = \frac{4}{3} \pi r^3$

Surface area of a sphere:  $A = 4\pi r^2$

Volume of a cylinder:  $V = \pi r^2 h$

Volume of a cone:  $V = \frac{1}{3} \pi r^2 h$

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1. [9 points] When a rocket leaves the gravitational influence of the Earth, it could travel infinitely far away (if we ignore the effects of other celestial bodies). When a rocket of mass  $m$  kilograms is  $h$  meters above the surface of the Earth, it has a weight of  $w = 9.8m \left( \frac{6,400,000}{6,400,000+h} \right)^2$  Newtons. Here, 6,400,000 is the radius of the Earth in meters, and 9.8 is the gravitational constant in  $\text{m/s}^2$ .
- a. [3 points] Approximately how much work is required to lift the rocket  $\Delta h$  additional meters when it is already  $h$  meters above the surface of the Earth? Your answer may include  $m$ ,  $h$ , and  $\Delta h$ .
- b. [6 points] Figure out the work required to lift the rocket from the surface of the Earth to a height of infinity. Your answer may include  $m$ .

2. [18 points] For each of the following series, write whether the series “Converges” or “Diverges” on the space provided next to the series. Support your answer by stating the test(s) you used to prove convergence or divergence, and show complete work and justification.

a. [6 points]  $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^2-1}$  \_\_\_\_\_

b. [6 points]  $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$  \_\_\_\_\_

c. [6 points]  $\sum_{n=2}^{\infty} \frac{\sin(n)}{n^2-3}$  \_\_\_\_\_

3. [12 points] For each of the following series, determine the interval of convergence and write it on the space provided to the right of the series. Be sure to show all appropriate work to justify your answer.

a. [6 points]  $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n}$  \_\_\_\_\_

b. [6 points]  $\sum_{n=1}^{\infty} \frac{n! x^n}{n^{10}}$  \_\_\_\_\_

4. [8 points] Consider a solid whose base is contained between the curves  $y = e^x$ ,  $y = 1$ , and  $x = 3$ . Cross-sectional slices perpendicular to the x-axis are rectangles, having length contained in the base region mentioned above and height determined by  $g(x) = x^2$ . Determine the exact volume of this solid.

5. [8 points] Kyle buys a nine-ounce cup of coffee at the store 10 minutes before class every morning. He doesn't drink the coffee until he arrives at class, ten minutes later. Suppose the coffee's temperature is initially  $180^\circ\text{F}$  when he buys it.
- a. [2 points] The temperature of the coffee in degrees Fahrenheit,  $T$ , at  $t$  minutes after it was purchased satisfies the differential equation  $\frac{dT}{dt} = k(T - 70)$ , where  $k$  is a constant and  $70^\circ\text{F}$  is the surrounding air's temperature. Determine the units of  $k$ .
- b. [4 points] Solve the differential equation given in part (a), finding  $T$  as a function of  $t$ . Your answer may include the constant  $k$ .
- c. [2 points] Suppose that Kyle adds one ounce of  $40^\circ\text{F}$  milk upon arriving at class, ten minutes after the coffee was purchased. When one ounce of milk is mixed with nine ounces of coffee, the resulting mixture has the temperature equal to

$$\frac{1}{10}[(\text{Temp. of Milk}) + 9(\text{Temp. of Black Coffee})].$$

Determine the *exact* value of  $k$  if the temperature of the coffee-milk mixture is  $100^\circ\text{F}$ , immediately after the milk has been added.

6. [10 points] Consider the function  $f(x) = \ln(1 + x)$  and its Taylor series about  $x = 0$ .
- a. [4 points] Determine the first four non-zero terms of the Taylor series for  $f(x) = \ln(1 + x)$  about  $x = 0$ . Be sure to show enough work to support your answer.

- b. [4 points] Find the first three non-zero terms of the Taylor series for  $g(x) = \ln\left(\frac{1+x}{1-x}\right)$  about  $x = 0$ . Be sure to show enough work to support your answer. (*Hint: You may find it helpful to utilize properties of logarithms.*)

- c. [2 points] Find the exact value of the sum of the series

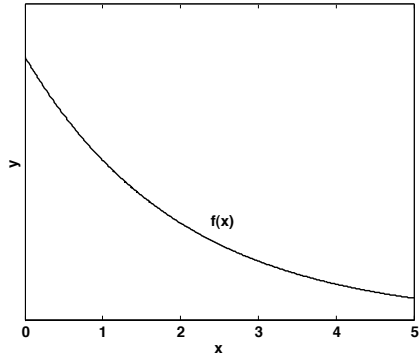
$$2 \left(\frac{3}{4}\right) + \frac{2}{3} \left(\frac{3}{4}\right)^3 + \frac{2}{5} \left(\frac{3}{4}\right)^5 + \dots$$



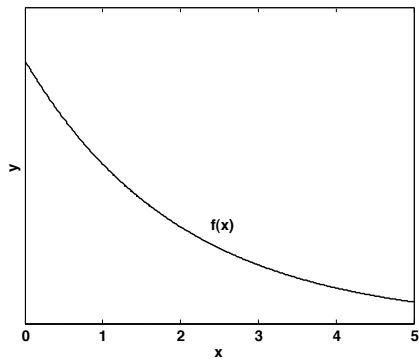
7. [11 points] Consider the differential equation  $\frac{dy}{dx} = \frac{1}{3^{10x}}$ .
- a. [5 points] Let  $C > 0$  be a small constant. Use Euler's method with step size  $\Delta x = C$  to write an expression for  $y(4C)$  given that  $y(0) = 7.3$ . Your answer may include  $C$ .
- b. [3 points] Write a closed form expression for the approximation of  $y(nC)$ , where  $n$  is a positive integer.
- c. [3 points] Approximate  $y(30)$  using  $C = 0.1$ .

8. [12 points] For each of the following questions, draw a visual interpretation on the provided graph that would be useful in determining the specified quantity. Then write one complete sentence to explain your sketch.

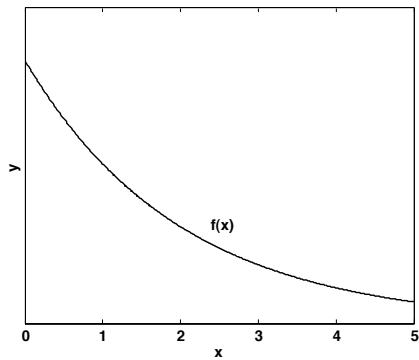
a. [4 points]  $\frac{f(4)-f(1)}{4-1}$



b. [4 points]  $\frac{1}{3} \int_1^4 f(x) dx$



c. [4 points]  $(f(2) + f(3) + f(4))$



9. [12 points] For each statement below, circle TRUE if the statement is *always* true; otherwise, circle FALSE. There is no partial credit on this page.

a. [2 points] If the power series  $\sum C_n x^n$  converges at  $x = 1$ , then it converges at  $x = -1$ .

True                  False

b. [2 points] Consider the point  $(x_0, y_0)$  given in Cartesian coordinates and its polar coordinates equivalent  $(r_0, \theta_0)$ . If  $\frac{y_0}{x_0} = 1$ , then  $\theta_0 = \frac{\pi}{4}$ .

True                  False

c. [2 points]  $\frac{d}{dx} \left( \int_x^2 \cos(\sin(t^2)) dt \right) = \cos(\sin(x^2))$ .

True                  False

d. [2 points] Suppose  $h(x)$  is a continuous function for  $x > 0$ . If  $\int_1^\infty h(x) dx$  converges then for constant  $0 < a < 1$ ,  $\int_1^\infty h\left(\frac{x}{a}\right) dx$  also converges.

True                  False

e. [2 points] If  $p(x)$  is a probability density function, then the units of  $\int_{-\infty}^\infty xp(x) dx$  are the same as the units of  $x$ .

True                  False

f. [2 points] The function  $P(x) = (x - 1) - \frac{1}{3!}(x - 1)^3$  is the third degree Taylor polynomial for  $f(x) = \sin(\pi x)$  about  $x = 1$ .

True                  False