Math 116 — First Midterm

February 8, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	11	
2	13	
3	8	
4	12	
5	10	
6	12	
7	7	
8	6	
9	6	
10	15	
Total	100	

1. [11 points] There is a classic result in mathematics, which states that the number of prime numbers less than any number $x \geq 2$ is approximated by the function $\text{li}(x) = \int_2^x \frac{dt}{\ln t}$.

a. [3 points] Is $\text{li}(x)$ increasing, decreasing, or neither for $x \geq 2$? Provide justification for your answer.

Solution: This function is increasing. To see that it is increasing, we take its derivative

$$\frac{d}{dx}\text{li}(x) = \frac{d}{dx}\int_2^x \frac{dt}{\ln t} = \frac{1}{\ln x} > 0.$$

-OR-

Since $\ln x$ is positive for $x \geq 2$, we know $f(x) = \frac{1}{\ln x} > 0$. Since $f(x)$ is the derivative of $\text{li}(x)$, we know $\text{li}(x)$ is increasing.

b. [3 points] Is $\text{li}(x)$ concave up, concave down, or neither for $x \geq 2$? Provide justification for your answer.

Solution: This function is concave down. To see it is concave down, take the second derivative:

$$\frac{d^2}{dx^2}\text{li}(x) = \frac{d}{dx}\frac{1}{\ln x} = -\frac{1}{x\ln^2 x} < 0.$$

-OR-

Since $\ln x$ is an increasing function, we know $f(x) = \frac{1}{\ln x}$ is decreasing, which means $f'(x) < 0$. Since $f'(x)$ is the second derivative of $\text{li}(x)$, we know $\text{li}(x)$ is concave down.

c. [5 points] Using Integration by Parts, put $\text{li}(x)$ into the form

$$\text{li}(x) = f(x) + \int_2^x \frac{dt}{(\ln t)^2}.$$

What is $f(x)$?

Solution: The way to get the second integral from the first looks like integration by parts. So we integrate $\int_2^x \frac{dt}{\ln t}$ by parts with $u = \frac{1}{\ln t}$ and $dv = dt$, giving $du = -\frac{dt}{t(\ln t)^2}$ and $v = t$. We get

$$t\frac{1}{\ln t}\Big|_2^x + \int_2^x \frac{dt}{\ln^2 t} = \frac{x}{\ln x} - \frac{2}{\ln 2} + \int_2^x \frac{dt}{\ln^2 t}.$$

So $f(x) = \frac{x}{\ln x} - \frac{2}{\ln 2}$.

2. [13 points] An ice cream cone has a height of 15 centimeters and the diameter of the top is 5 centimeters. The cone is filled with soft-serve ice cream such that the ice cream completely fills the cone, but does not exceed the top of the cone. The ice cream has a constant density of 2 grams per cubic centimeter.

a. [5 points] Write an expression for the approximate mass of ice cream contained in a circular cross-sectional slice that is located h_i centimeters from the the bottom tip of the cone and has depth Δh centimeters . Your answer may be in terms of h_i and Δh. Don't forget to include units.

Solution: Using similar triangles, at h_i centimeters from the tip of the cone, the radius is $\frac{1}{6}h_i$. This means the volume of a slice of depth Δh at height h_i is approximately $\frac{\pi}{36}h_i^2\Delta h$. Therefore, the mass of the slice is volume times density, which gives $\frac{2\pi}{36}h_i^2\Delta h = \frac{\pi}{18}h_i^2\Delta h$ grams.

b. [4 points] Set up a definite integral that can be used to determine the EXACT total mass of ice cream that is filling the cone, then solve for this exact value. Include appropriate units in your answer.

Solution: By summing up all such slices as found in part (a) and letting $\Delta h \to 0$, we have

$$\text{Total Mass} = \int_0^{15} \frac{\pi}{18}h^2 dh = \frac{\pi}{54}h^3|_0^{15} = \frac{3375\pi}{54} = \frac{125\pi}{2}\text{grams.}$$

c. [4 points] At what height above the tip of the cone is the center of mass of the ice cream? Give an EXACT answer, show all work, and include appropriate units.

Solution: We already found the total mass in part (b), but we need to calculate the total moments. We have

$$\int_0^{15} \frac{\pi}{18}h^3 dh = \frac{\pi}{72}h^4|_0^{15} = \frac{50625\pi}{72} = \frac{5625\pi}{8}.$$

Putting this together, we get that

$$\overline{h} = \frac{\frac{5625\pi}{8}}{\frac{125\pi}{2}} = \frac{45}{4} \text{ cm from the bottom of the cone.}$$

3. [8 points] When a spaceship takes off it does not travel in a straight path as it ascends. Instead, it turns slightly east, so that it gains speed by traveling with the rotation of the earth. From mission control's point of view, the spaceship's path appears to follow the curve $y = \sqrt{1+10x^2} - 1$, where y is the height in meters of the spaceship off the ground and x is the horizontal movement in meters from the launch pad. After 20 seconds, the spaceship appears to be 1 kilometer high.

a. [2 points] Determine, to the nearest hundredth of a meter, the horizontal distance the spaceship has traveled from the launch pad at 20 seconds.

Solution: When $y = 1000$ we solve for x:

$$\begin{aligned} 1000 =& \sqrt{1+10x^2} - 1 \\ 1001^2 =& 1 + 10x^2 \\ 1002000 =& 10x^2 \\ x =& \sqrt{100200} \approx 316.54 \text{ meters} \end{aligned}$$

b. [6 points] From mission control's point of view, what is the total distance of the path the spaceship appears to have traveled throughout the first 20 seconds of its trip? Give your answer to the nearest hundredth of a meter, and be sure to show enough work to justify your answer.

Solution: When $y = 0$ we can also see that $x = 0$. Now we use the formula to calculate arc length, noting that $\frac{dy}{dx} = \frac{10x}{\sqrt{1+10x^2}}$, and we get

$$\int_0^{316.543} \sqrt{1 + \frac{100x^2}{1+10x^2}}\,dx \approx 1048.95 \text{ meters},$$

by using the calculator to determine this integral.

4. [12 points] The function $f(x) = \int_0^x 10e^{-t^2} dt$ appears frequently in statistical analysis.

a. [6 points] Without calculating them, order $\int_0^2 f(x)dx$, MID(4), and TRAP(4) from smallest to biggest, where MID(4) and TRAP(4) are approximations for $\int_0^2 f(x)dx$. Show all work to justify your answer.

Solution: Since $f(x)$ is concave down, $\text{TRAP}(4) \leq \int_0^2 f(x)dx \leq \text{MID}(4)$. We can calculate the concavity of the function $f(x)$ by taking its second derivative:

$$\frac{d^2}{dx^2} f(x) = \frac{d^2}{dx^2} \int_0^x 10e^{-t^2} dt = \frac{d}{dx} 10e^{-x^2} = -20xe^{-x^2} < 0.$$

-OR-

The function $g(x) = 10e^{-x^2}$ is positive and decreasing for $x \geq 0$, so we know $g'(x) < 0$ on this interval. Since $g(x)$ is the derivative of $f(x)$, $g'(x)$ is the second derivative of $f(x)$, and we know that $f(x)$ is concave down, which gives $\text{TRAP}(4) \leq \int_0^2 f(x)dx \leq \text{MID}(4)$.

b. [2 points] Consider the following table, which evaluates $f(x) = \int_0^x 10e^{-t^2} dt$ for the specified values of x.

x	0	0.5	1	1.5	2
$f(x)$	A	4.613	7.468	8.562	B

What are the values of A and B? Write your answers on the spaces provided, rounding to three decimal places.

A = **0** B = **8.821**

c. [4 points] Using the table provided in part (b) and the answers you found in part (b), calculate LEFT(4) and RIGHT(4) to estimate the integral $\int_0^2 f(x)dx$. Be sure to show enough work to support your answer.

Solution: For four subintervals on the interval $0 \leq x \leq 2$, we need $\Delta x = 0.5$.

$$\text{LEFT}(4) = 0.5(0 + 4.613 + 7.468 + 8.562) = 10.3215$$

$$\text{RIGHT}(4) = 0.5(4.613 + 7.468 + 8.562 + 8.821) = 14.732$$

5. [10 points] For each statement below, circle TRUE if the statement is *always* true; otherwise, circle FALSE. There is no partial credit on this page.

a. [2 points] The function $\frac{\sin x}{x}$ has an anti-derivative.

[True] False

b. [2 points] $\frac{d}{dx}\int_x^{x^2} e^{t^2}dt = 4x^3e^{x^4} - 2xe^{x^2}$.

True [False]

c. [2 points] The average of the function $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 3$ is $\ln(\sqrt{3})$.

[True] False

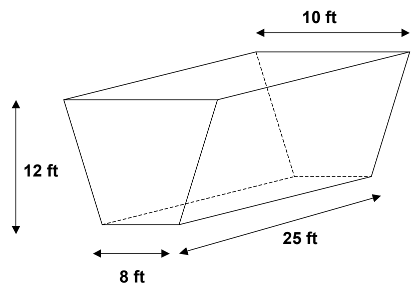
d. [2 points] $\int_a^b f(x)dx$ is greater than or equal to at least one of LEFT(n), RIGHT(n), TRAP(n), or MID(n) regardless of what $f(x)$ or n is.

True [False]

e. [2 points] If $\int_a^b f(x)dx > 0$ then $f(b) > f(a)$.

True [False]

6. [12 points] The dimensions of a large in-ground reservoir are shown in the figure below. (The ends of the reservoir are trapezoids.) The top of the reservoir is at ground level. Currently, water fills the bottom 9 feet of the reservoir. Recall that the density of water is 62.4 lb/ft^3.



a. [6 points] Write an expression that approximates the work done in lifting a horizontal slice of water that is y_i feet below ground level to the ground's surface, given that the depth of the slice is Δy. Include appropriate units in your answer.

Solution: A horizontal slice has length 25 ft, depth Δy ft, and the width is $10 - \frac{1}{6}y_i$ ft, so the volume is volume $= 25(10 - \frac{1}{6}y_i)\Delta y$. The weight of the slice is weight $= 1560(10 - \frac{1}{6}y_i)\Delta y$. The slice must be lifted a distance of y_i feet, so the work on the slice is given by

$$\text{work } = 1560y_i(10 - \frac{1}{6}y_i)\Delta y \text{ ft-lbs.}$$

b. [6 points] How much work is done to pump all of the water currently in the reservoir to the ground's surface? Be sure to include units and show enough work to support your answer.

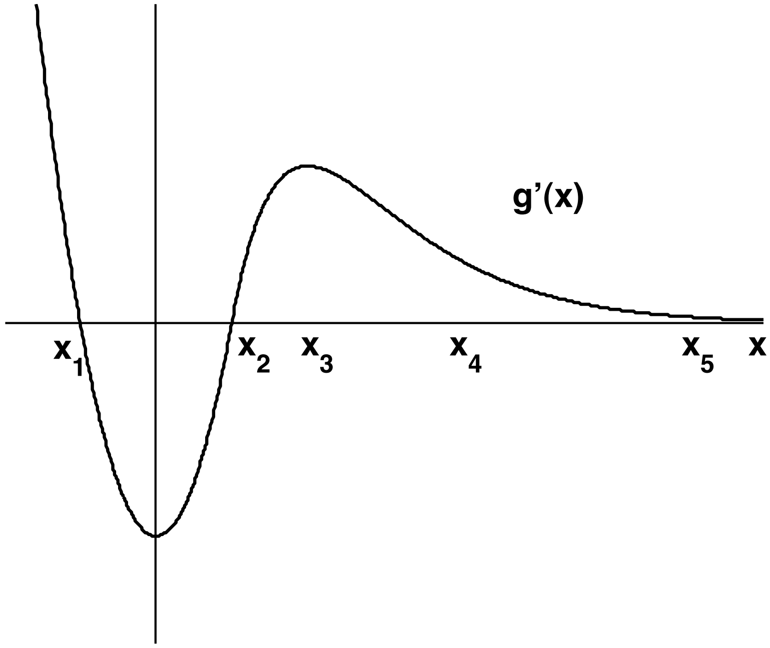
Solution: Summing up the slices that we found in part (a), we have

$$\text{total work } \approx \sum_{i=1}^{n} 1560y_i(10 - \frac{1}{6}y_i)\Delta y.$$

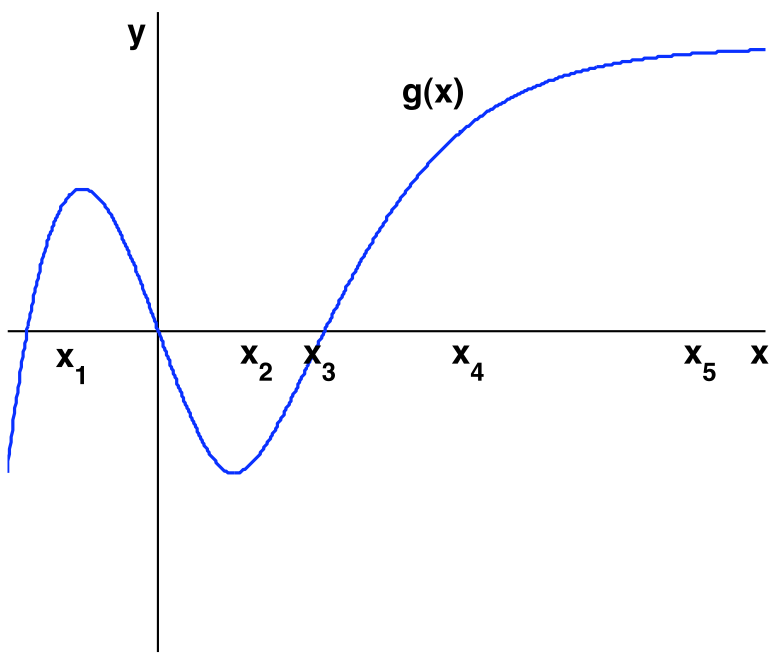
Letting $n \to \infty$, we have

$$\text{total work } = \int_3^{12} 1560y(10 - \frac{1}{6}y)dy = 905,580 \text{ ft-lbs.}$$

7. [7 points] Given is a graph of $g'(x)$. Sketch a graph of $g(x)$ on the provided axes given that $g(0) = 0$ and $g(x)$ is continuous. On your graph, label any local maxima, minima, and points of inflection.



Solution:



8. [6 points] Suppose that $\int_{-3}^{8} f(x)dx = 5$. Use this information to determine the values for the constants $a, b,$ and k that you are certain will satisfy the definite integral $\int_a^b kf(2x)dx = 5$. Write your answers on the spaces provided. You do not need to show your work for this problem.

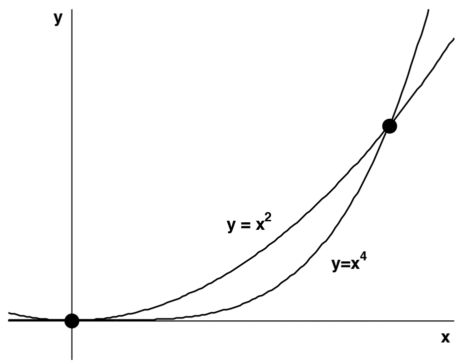
$a =$ **-1.5** $b =$ **4** $k =$ **2**

9. [6 points] Suppose $f(x) = f'(x) + 3$. Determine the EXACT value of $\int_0^1 e^x f'(x)dx$ given that $f(0) = 1$ and $f(1) = 4$. Be sure to show enough work to support your answer.

Solution: We use integration by parts, letting $u = e^x$ and $dv = f'(x)dx$ so that $du = e^x dx$ and $v = f(x)$. Then we have

$$\begin{aligned}
\int_0^1 e^x f'(x)dx &= e^x f(x)|_0^1 - \int_0^1 e^x f(x)dx \\
&= ef(1) - f(0) - \int_0^1 e^x(f'(x) + 3)dx \\
&= 4e - 1 - \int_0^1 e^x f'(x)dx - 3\int_0^1 e^x dx \\
2\int_0^1 e^x f'(x) &= 4e - 1 - 3e^x|_0^1 \\
\int_0^1 e^x f'(x) &= \frac{1}{2}((4e-1) - (3e-3)) = \frac{e+2}{2}
\end{aligned}$$

10. [15 points] Consider the area between the curves $y = x^2$ and $y = x^4$ in the positive quadrant as shown in the graph below. Use this area to answer the following questions.



a. [5 points] Set up, but do not evaluate, a definite integral that describes the area described above. Write your final answer on the space provided.

$\int_0^1 (x^2 - x^4)dx$ **or** $\int_0^1 (y^{1/4} - y^{1/2})dy$

b. [5 points] Set up, but do not evaluate, a definite integral that describes the volume of the solid generated by revolving the area described above about the line $y = 2$. Write your final answer on the space provided.

$\int_0^1 \pi((2 - x^4)^2 - (2 - x^2)^2)dx$

c. [5 points] Set up, but do not evaluate, a definite integral that describes the volume of the solid whose base is the area described above and whose cross-sections perpendicular to the x-axis are squares.

$\int_0^1 (x^2 - x^4)^2 dx$