Math 116 — Second Midterm

March 23, 2010

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	10	
3	6	
4	12	
5	10	
6	12	
7	12	
8	14	
9	14	
Total	100	

1. [10 points] There is a bucket, shaped like a cylinder, with a radius of 5 inches and a height of 20 inches. It has a circular hole in the bottom which has a radius of 1 inch. The bucket begins full of water, but it flows out the hole in the bottom. Let t be the number of seconds since the water began dripping from the bucket, and let $V(t)$ denote the volume (in inches^3) of water remaining in the bucket at time t. Let $h(t)$ be the depth of the water in the bucket at time t.

a. [2 points] Write a formula for the volume of water in the bucket, $V(t)$, as a function of the depth of the water in the bucket, $h(t)$.

Solution:

$$V(t) = 25\pi h(t)$$

b. [8 points] The volume of water changes such that it satisfies the differential equation

$$\frac{dV}{dt} = -0.6\pi\sqrt{19.6h}.$$

Solve for the depth of the water at time $t = 10$. Be sure to include units in your answer.

Solution: We know that $V = 25\pi h$, so $h = \frac{V}{25\pi}$. Substituting this we get that $\frac{dV}{dt} = -0.6\pi\sqrt{\frac{19.6V}{25\pi}} = -0.6\sqrt{.784\pi V}$. Using separation of variables, we have

$$\begin{aligned}
\frac{dV}{V^{1/2}} &= -0.6\sqrt{.784\pi}dt \approx -0.9416dt \\
2V^{1/2} &= -0.6\sqrt{.784\pi}t + C \\
V^{1/2} &= -0.3\sqrt{.784\pi}t + C \\
V &= (-0.3\sqrt{.784\pi}t + C)^2 \approx (-0.4708t + C)^2
\end{aligned}$$

When $t = 0$, $V = 500\pi$, so $V = 500\pi = C^2$, giving $C = \sqrt{500\pi} \approx 39.6333$, so $V = (-0.3\sqrt{.784\pi}t + \sqrt{500\pi})^2 \approx (-0.4708t + 39.6333)^2$. When $t = 10$, $V = 1219.7747$ inches ^{3}, so $h \approx 15.5307$ inches.

2. [10 points] Determine if each of the following integrals diverges or converges. If the integral converges, find the exact answer. If the integral diverges, write "DIVERGES." Show ALL work and use proper notation. Calculator approximations will not receive credit.

a. [5 points] $\int_0^2 \frac{3}{x^{1/3}} dx$

Solution:

$$\begin{aligned}\int_0^2 \frac{3}{x^{1/3}} dx &= \lim_{a\to 0^+} \int_a^2 \frac{3}{x^{1/3}} dx \\ &= \lim_{a\to 0^+} \frac{9}{2} x^{2/3}|_a^2 \\ &= \lim_{a\to 0^+} \left(\frac{9}{2}(2)^{2/3} - \frac{9}{2} a^{2/3}\right) \\ &= \frac{9}{2}(2)^{2/3}.\end{aligned}$$

The integral converges to $\frac{9}{2}(2)^{2/3}$.

b. [5 points] $\int_0^2 \frac{e^{-1/x}}{x^2} dx$

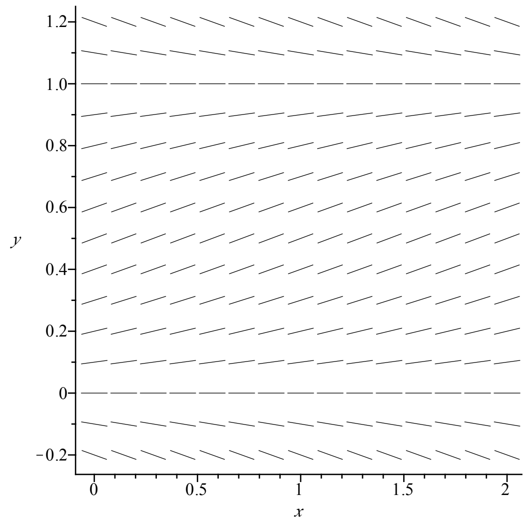
Solution: Make the substitution $u = -\frac{1}{x}$, so $du = \frac{1}{x^2} dx$. Then we have the general antiderivative $\int \frac{e^{-1/x}}{x^2} dx = \int e^u du = e^u + C = e^{-1/x} + C$. That gives us

$$\begin{aligned}\int_0^2 \frac{e^{-1/x}}{x^2} dx &= \lim_{a\to 0^+} \int_a^2 \frac{e^{-1/x}}{x^2} dx \\ &= \lim_{a\to 0^+} e^{-1/x}|_a^2 \\ &= \lim_{a\to 0^+} \left(e^{-1/2} - e^{-1/a}\right) \\ &= e^{-1/2}.\end{aligned}$$

The integral converges to $e^{-1/2}$.

3. [6 points]

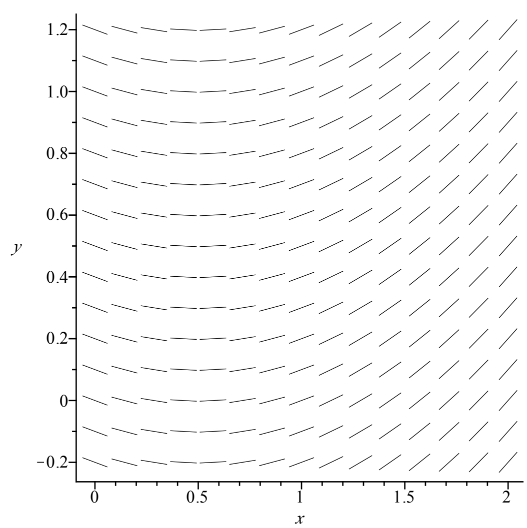
a. [3 points] The following figure is the slope field for $\frac{dy}{dx} = ay(b-y)$, where a and b are constants.



Which of the following statements is true regarding a and b? Circle only one answer.

A. $\boxed{a > 0, b > 0}$ B. $a > 0, b < 0$ C. $a < 0, b < 0$ D. $a < 0, b > 0$

b. [3 points] The following figure is the slope field for $\frac{dy}{dx} = ax + b$, where a and b are constants.



Which of the following statements is true regarding a and b? Circle only one answer.

A. $a > 0, b > 0$ B. $\boxed{a > 0, b < 0}$ C. $a < 0, b < 0$ D. $a < 0, b > 0$

4. [12 points] A bank account earns 2.5% annual interest compounded continuously. Continuous payments are made out of the account at a rate of \$15,000 per year for 18 years.

a. [4 points] Write a differential equation describing the balance $B = f(t)$, where t is in years satisfying $0 \le t \le 18$.

Solution:

$$\frac{dB}{dt} = .025B - 15,000 = .025(B - 600,000)$$

b. [4 points] Solve the differential equation you found in part (a) given an initial balance of B_0.

Solution:

$$\begin{aligned} \frac{dB}{dt} &= .025(B - 600,000) \\ \int \frac{dB}{(B - 600,000)} &= \int .025dt \\ \ln|B - 600,000| &= .025t + C \\ B - 600,000 &= Ae^{.025t} \\ B &= Ae^{.025t} + 600,000 \end{aligned}$$

Given $B = B_0$ when $t = 0$, we have $B_0 = A + 600,000$, so $A = B_0 - 600,000$, giving $B = (B_0 - 600,000)e^{.025t} + 600,000$.

c. [4 points] What was the initial balance if the account has \$10,000 remaining 18 years after the account was opened? Give your answer to the nearest penny.

Solution: Solving for B_0 given that $B = 10,000$ when $t = 18$, we have $10,000 = (B_0 - 600,000)e^{.025(18)} + 600,000$, giving $-590,000 = (B_0 - 600,000)e^{.45}$, which leads to $B_0 = -590,000e^{-.45} + 600,000 \approx 223,799.39$.
The initial balance would be approximately \$223,799.39.

5. [10 points] Let t be the number of minutes a student waits for the Bursley-Baits bus. For constants a and b, the probability density function giving the distribution of t is

$$p(t) = \begin{cases} 0 & \text{if } t < 0 \\ ae^{-bt} & \text{if } 0 \le t < \infty. \end{cases}$$

According to this density function, the mean waiting time for the bus is 8 minutes.

a. [6 points] Determine the exact values of the constants a and b. Answers supported only by calculator work will not receive full credit. Write your final answers on the spaces provided.

Solution: Because this is a density function, $\int_0^\infty ae^{-bt}dt = 1$. We have

$$\lim_{c\to\infty}\int_0^c ae^{-bt}dt = \lim_{c\to\infty} -\frac{a}{b}e^{-bt}|_0^c = \lim_{c\to\infty}\left[\frac{a}{b} - \frac{a}{b}e^{-bc}\right] = \frac{a}{b} = 1.$$

This gives us the condition $a = b$.
We know the mean time is 8, so we also have $8 = \int_0^\infty tae^{-bt}dt$. Then we have $8 = \lim_{c\to\infty}\int_0^c ate^{-bt}dt$. Use integration by parts using $u = at$, $du = adt$, $dv = e^{-bt}dt$, $v = -\frac{1}{b}e^{-bt}$. This gives us

$$\lim_{c\to\infty}\left(-\frac{a}{b}te^{-bt}|_0^c + \int_0^c \frac{a}{b}e^{-bt}dt\right) = \lim_{c\to\infty}\left(-\frac{a}{b}ce^{-bc} - \frac{a}{b^2}e^{-bt}|_0^c\right)$$

$$= \lim_{c\to\infty}\left(-\frac{a}{b}ce^{-bc} - \frac{a}{b^2}e^{-bc} + \frac{a}{b^2}\right) = \frac{a}{b^2} = 8.$$

Using $a = b$, we have $8 = \frac{b}{b^2} = \frac{1}{b}$, so that $b = \frac{1}{8} = a$.

$a =$ **1/8** $b =$ **1/8**

b. [4 points] Using your answers from part (a), determine the exact value for median waiting time. Include units in your answer. Answers supported only by calculator work will not receive full credit.

Solution: Let M be the median so that

$$0.5 = \int_0^M \frac{1}{8}e^{-t/8}dt.$$

Integrating we get

$$0.5 = -e^{-t/8}|_0^M = -e^{-M/8} + 1.$$

This gives us $e^{-M/8} = 0.5$, so that $-\frac{M}{8} = \ln(0.5)$, and $M = -8\ln(0.5) = 8\ln(2)$ minutes.

6. [12 points] The position of a particle at time t is given by $x = \cos\left(e^t\right)$, and $y = \cos\left(3e^t\right)$, where both x and y are measured in cm, and t is measured in seconds.

a. [5 points] Find the exact speed of the particle at time $t = 0$. Show enough work to support your answer and include units. Calculator approximations will not receive full credit.

Solution: We first find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = -e^t \sin(e^t) \text{ and } \frac{dy}{dt} = -3e^t \sin(3e^t)$$

$$\text{Speed } = \sqrt{(-e^t \sin(e^t))^2 + (-3e^t \sin(3e^t))^2} = \sqrt{e^{2t} \sin^2(e^t) + 9e^{2t} \sin^2(3e^t)}$$

Evaluating at $t = 0$ gives us Speed $= \sqrt{\sin^2(1) + 9\sin^2(3)}$ cm/sec.

b. [7 points] Use derivatives to determine the concavity of the particle's path at time $t = 0$.

Solution:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt}.$$

From part (a), $\frac{dy}{dx} = \frac{-3e^t \sin(3e^t)}{-e^t \sin(e^t)} = \frac{3\sin(3e^t)}{\sin(e^t)}$.

$$\frac{d}{dt}\left(\frac{3\sin(3e^t)}{\sin(e^t)}\right) / \frac{dx}{dt} = \frac{9e^t \cos(3e^t)\sin(e^t) - 3e^t \sin(3e^t)\cos(e^t)}{\sin^2(e^t)} / (-e^t \sin(e^t))$$

$$= \frac{9e^t \cos(3e^t)\sin(e^t) - 3e^t \sin(3e^t)\cos(e^t)}{-e^t \sin^3(e^t)}.$$

Evaluating at $t = 0$, we have $\frac{d^2y}{dx^2} = \frac{9\cos(3)\sin(1) - 3\sin(3)\cos(1)}{-\sin^3(1)} \approx 12.9673 > 0$. Therefore the path is concave up when $t = 0$.

7. [12 points] Suppose that functions $f(x)$, $g(x)$, and $h(x)$ are continuous and differentiable for $x \geq 1$ and satisfy the condition that $0 \leq f(x) \leq g(x) \leq h(x)$ for $x \geq 1$. Furthermore, suppose that $\int_1^\infty g(x)dx$ converges.

You do not need to show your work for this page. No partial credit will be given.

a. [4 points] Consider the following group of statements:

I. $\int_1^\infty h(x)dx$ diverges.

II. $\int_1^\infty h(x)dx$ converges.

III. $\int_1^3 h(x)dx$ converges.

Which of the above statements must be true? Circle ONE of the following choices:

A. I is true.

B. II is true.

C. III is true.

D. I and III are true.

E. II and III are true.

b. [4 points] Consider the following group of statements:

I. $\int_1^\infty f(x)dx$ diverges.

II. $\int_1^\infty f(x)dx$ converges.

III. $\int_1^3 f(x)dx$ converges.

Which of the above statements must be true? Circle ONE of the following choices:

A. I is true.

B. II is true.

C. III is true.

D. I and III are true.

E. II and III are true.

c. [4 points] Consider the following group of statements:

I. $\int_1^\infty (f(x) + g(x))dx$ converges.

II. $\int_1^\infty (h(x) + g(x))dx$ converges.

III. $\int_1^\infty \frac{g(x)}{x}dx$ converges.

Which of the above statements must be true? Circle ONE of the following choices:

A. I is true.

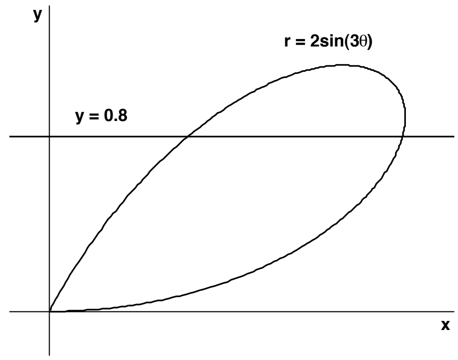
B. II is true.

C. III is true.

D. I and III are true.

E. II and III are true.

8. [14 points] Consider the area contained above the line $y = 0.8$, and below the curve $r = 2\sin(3\theta)$. You may find the following figure helpful.



a. [4 points] Find the (x, y) coordinates for the two points where $y = 0.8$ and $r = 2\sin(3\theta)$ intersect as shown in the figure above. Show enough work to support your answer.

Solution: We convert $y = 0.8$ into $r = \frac{0.8}{\sin\theta}$, then solve for $\frac{0.8}{\sin\theta} = 2\sin(3\theta)$, which gives $\theta \approx 0.4296, 0.8623$. The corresponding coordinates are $(1.7464, 0.8)$ and $(0.6854, 0.8)$.

b. [4 points] Write an expression for the area that is specified. You do not need to evaluate your expression.

Solution:

$$\text{Area} = \frac{1}{2}\int_{.4296}^{.8623} 4\sin^2(3\theta)d\theta - \frac{1}{2}\int_{.4296}^{.8623} \frac{0.64}{\sin^2(\theta)}d\theta$$

c. [6 points] Calculate the perimeter that surrounds the specified area. You may round your final answer to two decimal places.

Solution: The distance along the line $y = 0.8$, is $1.7464 - 0.6854 = 1.0610$. Along the curve $r = 2\sin(3\theta)$, $x = 2\sin(3\theta)\cos(\theta)$ and $y = 2\sin(3\theta)\sin(\theta)$. We use this to find

$$\frac{dx}{d\theta} = -2\sin(3\theta)\sin(\theta) + 6\cos(3\theta)\cos(\theta) \text{ and } \frac{dy}{d\theta} = 2\sin(3\theta)\cos(\theta) + 6\cos(3\theta)\sin(\theta).$$

$$\text{arc length} = \int_{.4296}^{.8623} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \approx 1.3737$$

The total perimeter is approximately $1.3737 + 1.0610 = 2.4347 \approx 2.43$.

9. [14 points] An ice cube melts at a rate proportional to its surface area. Let $V(t)$ denote the volume (in cm^3) of the ice cube, and let $x(t)$ denote the length (in cm) of a side of the ice cube t seconds after it begins to melt.

a. [4 points] Write a differential equation for $V(t)$, the ice cube's volume t seconds after it started melting. Your differential equation may contain V, t and an unknown constant k.

Solution: We know that $V = x^3$, so $x = V^{1/3}$. The surface area of the cube is given by $6x^2$. That gives us $\frac{dV}{dt} = 6kx^2$, and substituting x in terms of V, we have $\frac{dV}{dt} = 6kV^{2/3}$.

b. [4 points] The ice cube's initial volume is $V_0 > 0$. Solve the differential equation you found in part (a), finding V in terms of t, k, and V_0.

Solution: Using separation of variables, we have $\frac{dV}{V^{2/3}} = 6kdt$. This gives $3V^{1/3} = 6kt + C$, and $V^{1/3} = 2kt + C$, or $V = (2kt + C)^3$. When $t = 0$, $V = V_0$, which gives us $V_0 = C^3$, so that $C = V_0^{1/3}$. The solution is then $V = (2kt + V_0^{1/3})^3$.

c. [6 points] Graph the volume of the ice cube versus time given $V(0) = V_0$. Be sure to label your axes and any important features of your graph, including the time at which the ice cube has completely melted.

Solution: The vertical intercept is $V = V_0$. The horizontal intercept is $t = -\frac{1}{2k}V_0^{1/3}$.