Math 116 — First Midterm

February 6, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	14	
3	12	
4	10	
5	12	
6	11	
7	8	
8	10	
9	11	
Total	100	

1. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] If $\int_0^2 3f(x)+1\ dx = 8$, then $\int_0^2 f(x)\ dx = 2$.

True False

Solution:

$$\int_0^2 3f(x)+1\ dx = \int_0^2 3f(x)dx + \int_0^2 1dx = 3\int_0^2 f(x)dx + 2 = 8 \quad \text{then} \quad \int_0^2 f(x)\ dx = 2$$

b. [2 points] If $\int_a^b f(x)dx = 2$ and $\int_a^b g(x)dx = -3$ then $\int_a^b f(x)g(x)dx = -6$.

True **False**

Solution: For example: If $f(x) = 1$ and $g(x) = -\frac{3}{2}x$ with $a = 0$ and $b = 2$, then $\int_a^b f(x)dx = \int_0^2 dx = 2$ and $\int_a^b g(x)dx = \int_0^2 -\frac{3}{2}xdx = -3$.
But $\int_a^b f(x)g(x)dx = \int_0^2 -\frac{3}{2}xdx = -3 \neq -6$.

c. [2 points] If $f(x) = \int_{-2x}^0 \sqrt{1+t^4}dt$ then $f(x)$ is increasing.

True False

Solution: Since $f'(x) = -\sqrt{1+(-2x)^4}(-2) = 2\sqrt{1+16x^4} > 0$, then $f(x)$ is increasing.

d. [2 points] If $\int_0^1 f(x)dx \leq \int_0^1 g(x)dx$ then $f(x) \leq g(x)$ for $0 \leq x \leq 1$.

True **False**

Solution: For example: $f(x) = 1 - x$ and $g(x) = 2x$, then $\int_0^1 f(x)dx = \int_0^1 1 - xdx = \frac{1}{2}$ and $\int_0^1 g(x)dx = \int_0^1 2xdx = 2$. But $f(0) = 1 \geq g(0) = 0$.

e. [2 points] If $g(x)$ is odd and $\int_1^3 g(x)dx = 2$, then $\int_{-3}^1 g(x)dx = -2$.

True False

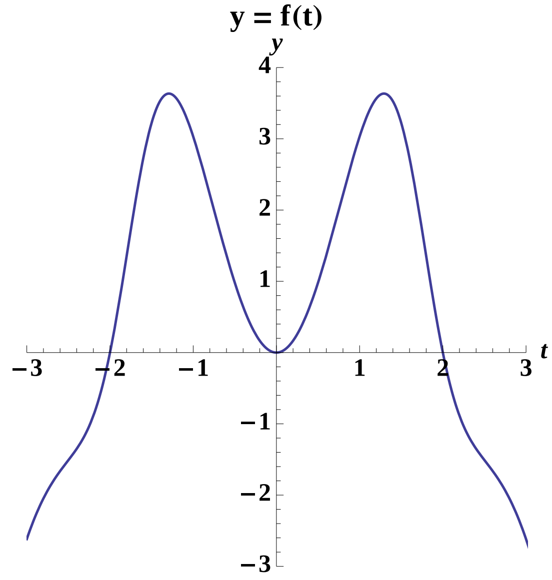
Solution: Since $g(x)$ is odd, then $\int_{-1}^1 g(x)dx = 0$ and $\int_{-b}^{-a} g(x)dx = -\int_a^b g(x)dx$. Hence $\int_{-3}^1 g(x)dx = \int_{-3}^{-1} g(x)dx + \int_{-1}^1 g(x)dx = \int_{-3}^{-1} g(x)dx = -\int_1^3 g(x)dx = -2$.

f. [2 points] If $f(t)$ is measured in dollars per year, and t is measured in years, then $\int_a^b f(t)dt$ is measured in dollars per years squared.

True **False**

Solution: The units for $\int_a^b f(t)dt$ are dollars (dollars per year (units for $f(t)$) times year (units for t)).

2. [14 points] Let $F(x) = \int_0^x f(t)dt$, where the graph of $f(t)$ is given below. In each blank space below, determine whether the number on the left is greater than, less than, or equal to the number to the right, and fill the blank with the symbol $>$, $<$, or $=$ accordingly. If there is not enough information to compare the given pair of numbers, write **none** in the blank space.



$F(-2)$ ________ $F(0)$ $\quad$ $F(-2)$ ________ $F(2)$

$F(2)$ ________ $F(3)$ $\quad$ $F(2)$ ________ 8

$F'(-2)$ ________ $F'(0)$ $\quad$ $F''(-2)$ ________ $F''(0)$

$\frac{1}{5}\int_{-2}^{3} f(t)dt$ ________ Average of $f(x)$ on $[0, 2]$.

Solution:

$F(-2) < F(0)$ $\quad$ $F(-2) < F(2)$

$F(2) > F(3)$ $\quad$ $F(2) < 8$

$F'(-2) = F'(0)$ $\quad$ $F''(-2) > F''(0)$

$\frac{1}{5}\int_{-2}^{3} f(t)dt <$ Average of $f(x)$ on $[0, 2]$.

3. [12 points] A boat travels in a straight line toward an island d km away. The velocity $v(t)$ (toward the island is positive velocity) in km/hr, t hours after departing from its starting position. The velocity $v(t)$ during the first three hours is recorded at half hour intervals, and is given in the table below:

t	0	0.5	1	1.5	2	2.5	3
$v(t)$	50	48	44	38	30	20	8

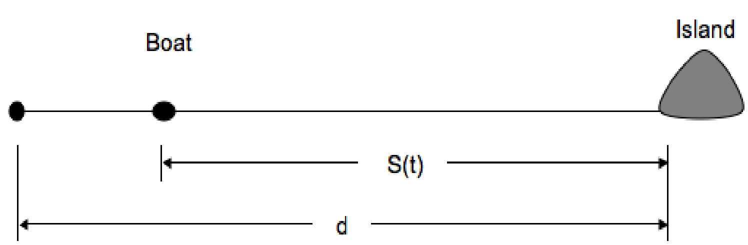
a. [8 points] Find an estimate for how far the boat is from the starting point after 3 hours using the four approximations LEFT, RIGHT, MID and TRAP. Use the maximum number of subintervals possible. Write each sum, and justify whether the sum is an underestimate or an overestimate. Assume the velocity is always decreasing and has no inflection points. **Circle your answers**.

Solution:

$$\text{LEFT}(6) = \frac{1}{2}(50+48+44+38+30+20) = 115 \qquad \text{overestimate.}$$
$$\text{RIGHT}(6) = \frac{1}{2}(48+44+38+30+20+8) = 94 \qquad \text{underestimate.}$$
$$\text{TRAP}(6) = \frac{1}{2}(115+94) = 104.5 \qquad \text{underestimate.}$$
$$\text{MID}(3) = (48+38+20) = 106 \qquad \text{overestimate.}$$

since $v(t)$ is decreasing and concave down.

b. [4 points] If it takes the boat 5 hours to reach the island, write an expression involving integrals for the distance $S(t)$ between the island and the boat for $0 \leq t \leq 5$.

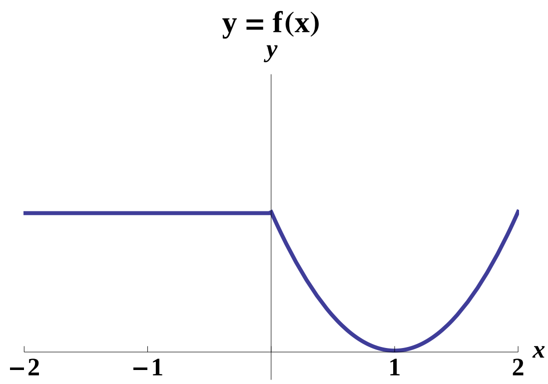


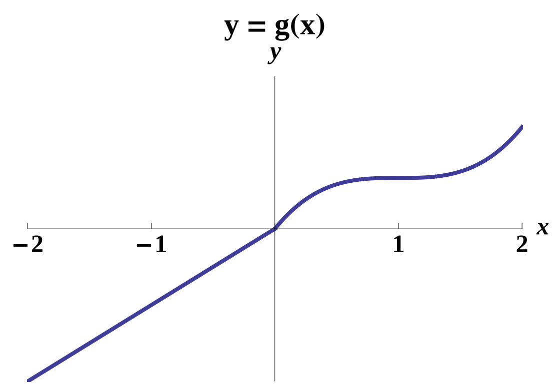
Solution:

$$S(t) = d - \int_0^t v(x)dx.$$

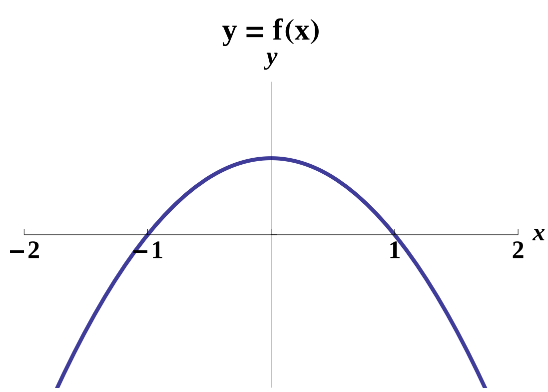
4. [10 points] Given the graph of $f(x)$, sketch the graph of $g(x)$. Make sure your graph accurately shows the intervals where $g(x)$ is increasing or decreasing and its concavity.

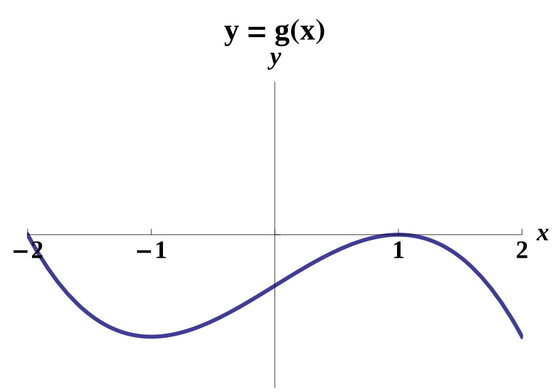
a. [5 points] Let $g'(x) = f(x)$ with $g(0) = 0$.



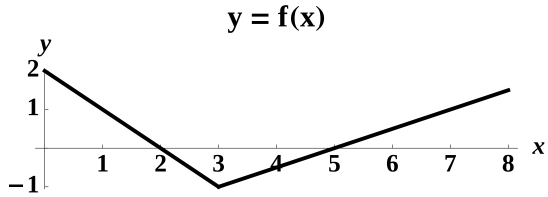


b. [5 points] Let $g(x) = \int_1^x f(t)dt$.





5. [12 points] The graph of $f(x)$ and a table of values for the continuous functions $g(x)$ and $h(x)$ are given below. The function $h(x)$ is an antiderivative of $g(x)$.



x	0	1	2	3	4
$g(x)$	1	3	5	7	9
$h(x)$	-3	-1	3	9	17

Compute the **exact** value of each of the following expressions:

a. [1 point] $\int_0^7 |f(x)|dx$

Solution: $\int_0^7 |f(x)|dx = 2 + \frac{3}{2} + 1 = \frac{9}{2}$.

b. [4 points] $\int_1^{e^2} \frac{f(\ln x)}{x}dx$

Solution: If $u = \ln x$, then

$$\int_1^{e^2} \frac{f(\ln x)}{x}dx = \int_0^2 f(u)du = 2$$

c. [7 points] Find $\int_1^2 xg'(2x)dx$

Solution: If $w = 2x$, then

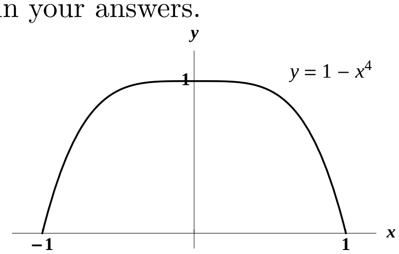
$$\int_1^2 xg'(2x)dx = \frac{1}{2}\int_2^4 \frac{w}{2}g'(w)dw = \frac{1}{4}\int_2^4 wg'(w)dw.$$

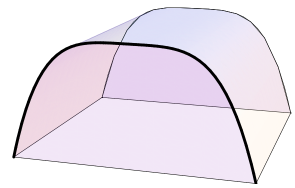
Integration by parts with $u = w$ and $v' = g'(w)$ yields

$$\begin{aligned}\int_2^4 wg'(w)dw &= wg(w)\,\big|_2^4 - \int_2^4 g(w)dw \\ &= 4g(4) - 2g(2) - (h(4) - h(2)) \\ &= 4(9) - 2(5) - (17 - 3) = 12.\end{aligned}$$

$$\int_1^2 xg'(2x)dx = \frac{1}{4}\int_2^4 wg'(w)dw = \frac{1}{4}(12) = 3.$$

6. [11 points] The lateral faces of a tank are determined by the curve $y = 1 - x^4$ and the x-axis (where x and y are measured in meters). The length of the tank is 10 meters. Be sure to include units in your answers.





a. [5 points] The tank is filled with water to a height of one half a meter. If the density of water is 1,000 kg/m^3, write an expression that approximates the mass of one slice of water y meters above the ground and Δy meters thick.

Solution: Mass is density times volume. The density is constant, and the volume of one slice at height y with thickness Δy is

$$20(1-y)^{1/4}\Delta y \quad \text{m}^3.$$

Therefore, the mass of the slice is

$$20,000(1-y)^{1/4}\Delta y \quad \text{kg}.$$

b. [2 points] Write a definite integral that represents the total mass of water in the tank.

Solution: We add up all of the slices from 0 to .5:

$$\int_0^{.5} 20,000(1-y)^{1/4}\,dy \quad \text{kg}.$$

c. [4 points] Write a definite integral that represents the amount of work required to pump the water to the top of the tank.

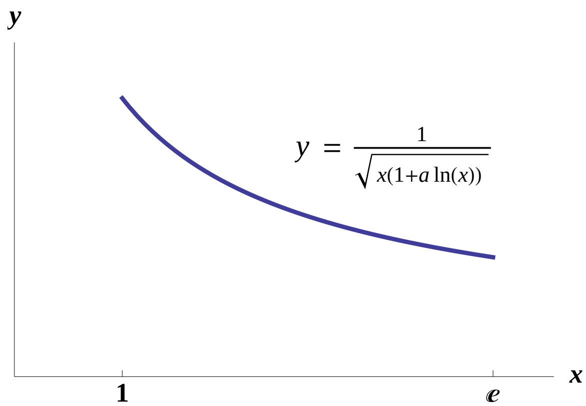
Solution: Work is force times distance. The force on a slice at height y with thickness Δy (or dy) is 9.8 times the mass, which we computed above. This slice travels a distance $(1-y)$ under this force. Therefore the work done on one slice is

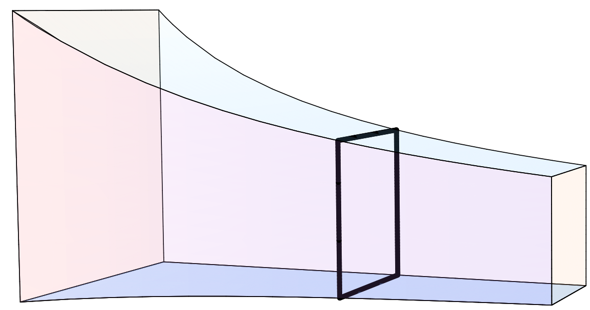
$$9.8 \cdot 20,000(1-y)^{1/4} \cdot (1-y)\Delta y \ \ \text{N}.$$

So we add up all of those slices to get

$$\int_0^{.5} 9.8 \cdot 20,000(1-y)^{1/4} \cdot (1-y)\,dy \ \ \text{J or N} \cdot \text{m}.$$

7. [8 points] Let S be the solid whose base is the region bounded by the graph of the curve $y = \frac{1}{\sqrt{x(1+a\ln(x))}}$ (for some positive constant $a > 0$), the x-axis, the lines $x = 1$ and $x = e$. The cross-sections of S perpendicular to the x-axis are squares. Find the exact volume of S. Show all your work to receive full credit.



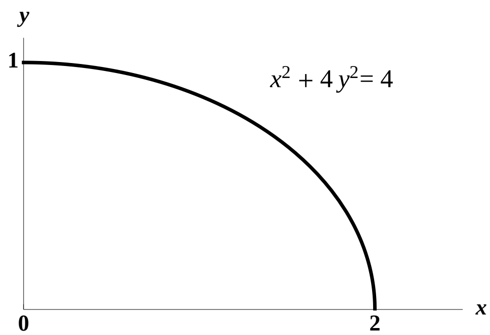


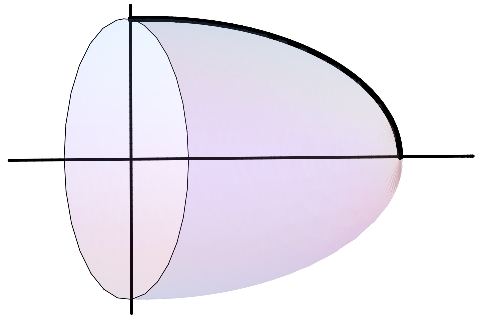
Solution: As the cross-section of each slice is a square with sidelength y, the volume of one slice is $y^2\Delta x$, Therefore the total volume is

$$\int_1^e \left(\frac{1}{\sqrt{x(1+a\ln x)}}\right)^2 dx = \int_1^e \frac{1}{x(1+a\ln x)}\,dx$$
$$= \frac{1}{a}\int_1^{1+a}\frac{1}{u}\,du$$
$$= \frac{1}{a}\ln|1+a|.$$

The second line comes from the u-substitution $u = 1 + a\ln x$.

8. [10 points] A bullet, with constant density $\delta = 3$ g/cm^3, has the shape of a solid generated by rotating the region enclosed by piece of the ellipse $x^2 + 4y^2 = 4$ lying in the first quadrant, the x-axis and the y-axis, around the x-axis. The variables x and y are measured in cm.





a. [5 points] Write an expression that approximates the mass of one slice of the bullet perpendicular to the x-axis at x cm from the y-axis and Δx cm thick.

Solution: Mass is density times volume. The slice at x is a right circular cylinder, so its mass is

$$3 \cdot \pi \left(\sqrt{1 - \tfrac{x^2}{4}}\right)^2 \Delta x \quad \text{grams.}$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the bullet's center of mass.

b. [3 points] Find a formula for $\bar{x}$.

Solution: The center of mass has x-coordinate

$$\bar{x} = \frac{\int_0^2 x\, dM}{M} = \frac{\int_0^2 3\pi x \left(1 - \frac{x^2}{4}\right)\, dx}{\int_0^2 3\pi \left(1 - \frac{x^2}{4}\right)\, dx}$$

c. [2 points] What can you say about $\bar{y}$. Justify.

Solution: Because of the rotational symmetry and constant density, we have $\bar{y} = 0$.

9. [11 points] In the following problems show all your work to receive full credit.

a. [7 points] The population of an invasive aquatic plant in a circular lagoon has density given by $\delta(r) = 20(1-e^{-r^2})$ kg/m^2, where r is the distance in meters from its center. The lagoon has radius R meters. Find the exact amount of plants living at the lake.

Solution: Since density is a function of radius, we will slice the biomass into rings with thickness Δr. The mass of one ring is approximately

$$\delta(r) \cdot 2\pi r \Delta r.$$

So now we add up all the slices, and we get total mass

$$\begin{aligned}
\int_0^R 20\left(1-e^{-r^2}\right) \cdot 2\pi r\ dr &= 20\pi \int_0^R 2r\left(1-e^{-r^2}\right)\ dr \\
&= 20\pi \int_0^{R^2} \left(1-e^{-u}\right)\ du \\
&= 20\pi \left(u+e^{-u}\right)\Big|_0^{R^2} \\
&= 20\pi(R^2+e^{-R^2}-1).
\end{aligned}$$

b. [4 points] Let

$$F(x) = \int_0^x \sqrt{e^{2t}-1}dt.$$

Find the exact value of the length of the curve on $0 \le x \le 1$.

Solution: The formula for length of an arc is

$$L = \int_0^1 \sqrt{1+(F'(x))^2}\ dx.$$

By the second FTC, $F'(x) = \sqrt{e^{2x}-1}$. Therefore the length of the arc is

$$\begin{aligned}
L = \int_0^1 \sqrt{1+(\sqrt{e^{2x}-1})^2}\ dx &= \int_0^1 \sqrt{1+(e^{2x}-1)}\ dx \\
&= \int_0^1 e^x\ dx \\
&= e-1.
\end{aligned}$$