Math 116 — Second Midterm

March 19, 2012

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	11	
3	10	
4	13	
5	13	
6	8	
7	9	
8	12	
9	14	
Total	100	

1. [10 points] Indicate if each of the following statements are true or false by circling the correct answer. **You do not need to justify your answers.**

a. [2 points] The integral $\int_{-2}^{2} \frac{1}{x^2} dx = -1$

True **[False]**

Solution: $\int_{-2}^{2} \frac{1}{x^2} dx = 2\int_{0}^{2} \frac{1}{x^2} dx = \lim_{b\to 0^+} 2\int_{b}^{2} \frac{1}{x^2} dx = \lim_{b\to 0^+} \frac{-2}{x} \Big|_b^2 = \infty$ diverges.

b. [2 points] For any positive number p, the integral $\int_{0}^{\infty} \frac{1}{x^p} dx$ diverges.

[True] False

Solution: $\int_{0}^{\infty} \frac{1}{x^p} dx = \int_{0}^{1} \frac{1}{x^p} dx + \int_{1}^{\infty} \frac{1}{x^p} dx$ The first integral diverges if $p \geq 1$ and the second diverges if $p \leq 1$. Hence the integral diverges for all values of p.

c. [2 points] If the median grade of an exam is larger than the average grade then more than half of the students got a grade greater or equal to the average.

[True] False

Solution: The median is the grade that divides the upper half of the grades from the lower half. If the average grades is lower than the median, then more than half of the students got a grade greater or equal to the average.

d. [2 points] Let $f(x)$ be a positive and continuous function. If $\lim_{x\to\infty} f(x) = \infty$, then $\int_{0}^{\infty} \frac{1}{f(x)} dx$ converges.

True **[False]**

Solution: Consider

$$f(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ x & \text{otherwise.} \end{cases}$$

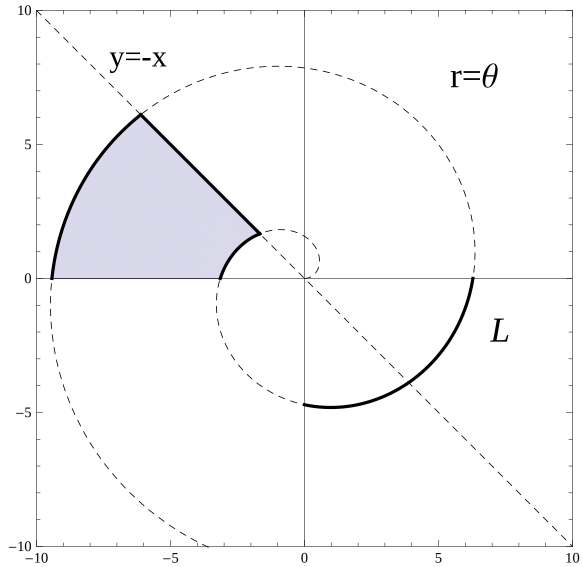
then $\int_{0}^{\infty} \frac{1}{f(x)} dx = \int_{0}^{1} 1 dx + \int_{1}^{\infty} \frac{1}{x} dx$ diverges since the second integral diverges.

e. [2 points] The line $y = 2x + 1$ has parametric equations $x = -1 + 2t$, $y = -1 + 4t$.

[True] False

Solution: $y = -1 + 4t = 2x + 1 = 2(-1 + 2t) + 1 = 4t - 1$

2. [11 points] Consider the graph of the spiral $r = \theta$ for $\theta \geq 0$.



In the following questions, write an expression involving definite integrals that computes the values of the following quantities (you do not need to evaluate any integrals) :

a. [4 points] The length of the arc L.

Solution: The arc length of a polar curve is given by the formula

$$\int_{\theta_1}^{\theta_2} \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} d\theta.$$

Therefore,

$$\text{Length of } L = \int_{3\pi/2}^{2\pi} \sqrt{\theta^2 + 1}\ d\theta.$$

b. [7 points] The area of the shaded region.

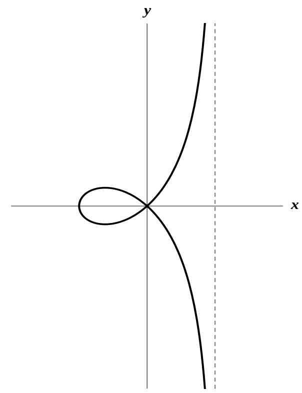
Solution: The of the region inside of a polar curve is

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} [r(\theta)]^2 d\theta.$$

We have to take the outer area minus the inner area. We can write the line $y = -x$ as $\theta = \frac{3\pi}{4} + 2\pi k$ for some k. The outer curve is on the second pass around the origin, and the inner curve is on the first time around the origin. Wo we get

$$A = \frac{1}{2}\int_{11\pi/4}^{3\pi} \theta^2\ d\theta - \frac{1}{2}\int_{3\pi/4}^{\pi} \theta^2\ d\theta.$$

3. [10 points] The motion of a particle is given by the following parametric equations



$$x(t) = \frac{a(t^2-1)}{t^2+1} \qquad\qquad y(t) = \frac{t^3-t}{t^2+1}.$$

for $-\infty < t < \infty$ and a positive constant a. Show all your work to receive full credit.

a. [3 points] Find the values of t at which the particle passes through the origin.

Solution: We need to solve simultaneously $x(t) = 0$ and $y(t) = 0$.

- $x(t) = 0$, then $\frac{a(t^2-1)}{t^2+1} = 0$. This is only possible if $t^2 - 1 = 0$. Hence $t = \pm 1$.
- $y(t) = 0$, then $\frac{t^3-t}{t^2+1} = 0$. This is only possible if $t^3 - t = 0$. Hence $t = 0, \pm 1$.

Times at the origin $t = \pm 1$.

b. [5 points] Find the value of t at which the curve defined by the parametric equations has a vertical tangent line. Also, give the (x, y) coordinates of this point.

Solution:

$$x'(t) = a\left[\frac{(t^2+1)(2t) - (t^2-1)(2t)}{(t^2+1)^2}\right] = \frac{4at}{(t^2+1)^2} \qquad \text{then} \quad x'(t) = 0 \qquad \text{at } t = 0.$$

and $(x(0), y(0)) = (-a, 0)$.

c. [2 points] The curve has a vertical asymptote. Find the equation of this asymptote.

Solution: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{a(t^2-1)}{t^2+1} = \lim_{t\to\infty} \frac{at^2}{t^2} = a$. Then the equation of the vertical asymptote is $x = a$.

4. [13 points]

a. [6 points] A cylindrical tank with height 8 m and radius of 8 m is standing on one of its circular ends. The tank is initially empty. Water is added at a rate of 2 m^3 /min. A valve at the bottom of the tank releases water at a rate proportional to the water's depth (proportionality constant $= k$). Let $V(t)$ be the volume of the water in the tank at time t, and $h(t)$ be the depth of the water at time t.

i. Find a formula for $V(t)$ in terms of $h(t)$. $V(t) =$ ________________________

ii. Find the differential equation satisfied by $V(t)$. Include initial conditions.

Solution: i) The formula is $V(t) = 64\pi h(t)$.

ii) The differential equation is

$$\frac{dV}{dt} = 2 - kh.$$

So now we can solve $h(t) = \frac{V(t)}{64\pi}$. Substituting in V for h, we get

$$\frac{dV}{dt} = 2 - k\frac{V}{64\pi}$$

with initial condition $V(0) = 0$.

b. [7 points] Let $M(t)$ be the balance in dollars in a bank account t years after the initial deposit. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = \frac{1}{100}M - a.$$

where a is a positive constant. Find a formula for $M(t)$ if the initial deposit is $1,000$ dollars. Your answer may depend on a.

Solution: This equation is separable:

$$\frac{dM}{M - 100a} = \frac{1}{100}dt.$$

Integrating, we find $\ln|M - 100a| = \frac{t}{100} + C$. So we get

$$M = Be^{t/100} + 100a.$$

Using the initial conditions, $M(0) = 1000$, so $1000 = B + 100a$. Substituting back in we get

$$M = 100\left((10 - a)e^{t/100} + a\right).$$

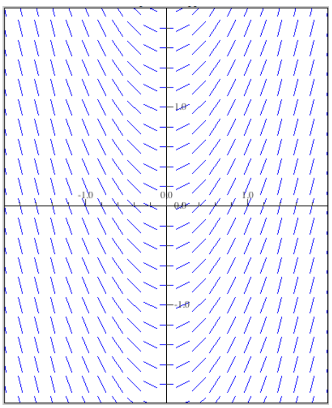
5. [13 points] Consider the following differential equations

A. $y' = 2x$ B. $y' = 5y - 1$ C. $yy' = 2$ D. $y' = \frac{y}{x}$

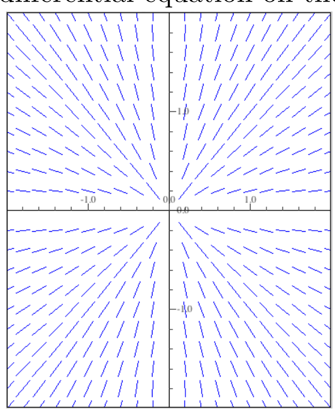
a. [6 points] Each of the following functions is a solution to one of the differential equations listed above. Indicate which differential equation with the corresponding letter (A,B,C or D) on the given line.

1. $y = \frac{1}{5} + e^{5x}$ ______B______
2. $y = x^2 + 1$ ______A______
3. $y = 2\sqrt{x}$ ______C______

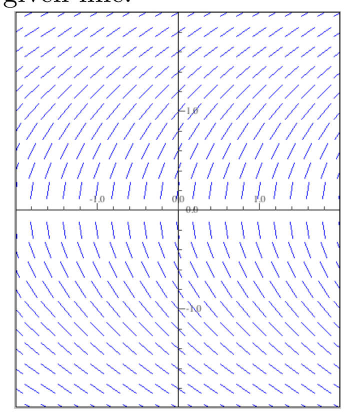
b. [3 points] Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line.



______A______



______D______



______C______

c. [4 points] Find the equilibrium solutions of the differential equations given above (if any). Write the equation of the equilibrium solutions in the space provided below. If the equation does not have equilibrium solutions, write none.

A. ___None___ B. ___$y = \frac{1}{5}$___ C. ___None___ D. ___$y = 0$___

6. [8 points] A box is dropped from an airplane. The downward velocity $v(t)$ of the box, once its parachute opens, satisfies the differential equation

$$\frac{dv}{dt} = 10 - \frac{1}{10}(1 + e^{-t})v^2.$$

Suppose the parachute opens when the velocity of the box is 11 m/s. Use Euler's method with three steps to approximate the velocity of the box one second after the parachute opens. Fill in the table below with the approximations at each step. Be sure to include all your work to receive full credit.

t	0			
$v(t)$				

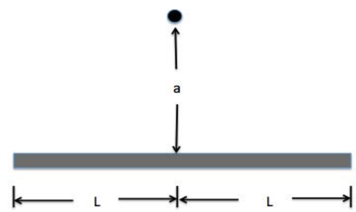
Solution: Recall Euler's method gives us

$$v_{i+1} = \left(\frac{dv}{dt}\bigg|_{x_i} \right) \cdot \Delta x + v_i.$$

Here $\Delta x = \frac{1}{3}$. So now we just compute the following table:

t	0	1/3	2/3	1
v	11	6.267	7.353	7.959
$\frac{dv}{dt}$	-14.2	3.259	1.8167	

7. [9 points] A particle of mass m is positioned at a perpendicular distance a from the center of a rod of length $2L$ and constant mass density δ as shown below



The force of gravitational attraction F between the rod and the particle is given by

$$F = Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx.$$

a. [5 points] Does the force of gravitational attraction F approach infinity as the length of the rod goes to infinity? Justify your answer using the comparison test.

Solution:

$$\begin{aligned}
F &= Gm\delta a \int_{-L}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx = 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
\lim_{L\to\infty} F &= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{L} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&= \lim_{L\to\infty} 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx \\
&\le 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + \lim_{L\to\infty} 2Gm\delta a \int_{1}^{L} \frac{1}{(x^2)^{\frac{3}{2}}} dx \\
&= 2Gm\delta a \int_{0}^{1} \frac{1}{(a^2+x^2)^{\frac{3}{2}}} dx + 2Gm\delta a \int_{1}^{\infty} \frac{1}{x^3} dx.
\end{aligned}$$

The last integral converges since $\int_1^\infty \frac{1}{x^p} dx$ converges for $p = 3 > 1$.

b. [4 points] Consider the integral

$$I = \int_1^\infty \frac{1}{(a^2+x^2)^p} dx$$

i. Give a power function which, if integrated over $[1, \infty)$, will have the same convergence or divergence behavior as I.

Solution: $\frac{1}{(x^2)^p} = \frac{1}{x^{2p}}$

ii. For which values of p would you predict I is convergent? For which would I be divergent?

Solution: Convergent: Need $2p > 1$, hence $p > \frac{1}{2}$.

Divergent: Need $2p \le 1$, hence $p \le \frac{1}{2}$.

8. [12 points] Determine if the following integrals converge or diverge. Justify your answer. If you use the comparison test, be sure to show all your work.

a. [3 points] $\displaystyle\int_1^\infty \frac{1}{x+e^x}dx.$

Solution: Using the comparison

$$\frac{1}{x+e^x} \leq \frac{1}{e^x},$$

we get convergence, since

$$\int_1^\infty \frac{1}{e^x}dx$$

converges.

b. [4 points] $\displaystyle\int_1^e \frac{1}{x(\ln x)^2}dx.$

Solution: This is improper because $\ln 1 = 0$, so there is an asymptote at $x = 1$. Here we use the substitution $u = \ln x$, so $du = \frac{1}{x}dx$, and we get

$$\int_1^e \frac{1}{x(\ln x)^2}dx = \int_0^1 \frac{1}{u^2}du.$$

The right hand side diverges by the p-test ($p = 2 > 1$).

c. [5 points] $\displaystyle\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx.$

Solution: Break this up into two integrals:

$$\int_{2\pi}^\infty \frac{x\cos^2 x + 1}{x^3}dx = \int_{2\pi}^\infty \frac{x\cos^2 x}{x^3}dx + \int_{2\pi}^\infty \frac{1}{x^3}dx$$

The second integral converges by the p-test. For the first, we need to use another comparison:

$$\frac{x\cos^2 x}{x^3} \leq \frac{1}{x^2}$$

so by comparison, the first integral also converges. The sum of two convergent improper integrals converges, so this integral converges.

9. [14 points] A machine produces copper wire, and occasionally there is a flaw at some point along the wire. The length x of wire produced between two consecutive flaws is a continuous variable with probability density function

$$f(x) = \begin{cases} c(1+x)^{-3} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show all your work in order to receive full credit.

a. [5 points] Find the value of c.

Solution: Since $f(x)$ is a density function $\int_{-\infty}^{\infty} f(x)dx = 1$. Then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} c(1+x)^{-3}dx = \lim_{b\to\infty} \int_{0}^{b} c(1+x)^{-3}dx$$
$$= \lim_{b\to\infty} \frac{-c}{2(1+x)^2}\Big|_0^b = \frac{c}{2} = 1$$

Hence $c = 2$.

b. [3 points] Find the cumulative distribution function $P(x)$ of the density function $f(x)$. Be sure to indicate the value of $P(x)$ for **all** values of x.

Solution:

$$P(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} c(1+t)^{-3}dt = \frac{-c}{2(1+t)^2}\Big|_0^x = \frac{c}{2} - \frac{c}{2(1+x)^2} = 1 - \frac{1}{(1+x)^2}.$$

$$P(x) = \begin{cases} 1 - \frac{1}{(1+x)^2} & x \geq 0. \\ 0 & x < 0. \end{cases}$$

c. [5 points] Find the mean length of wire between two consecutive flaws.

Solution:

$$\text{mean} = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} c \frac{x}{(1+x)^3} dx = c \lim_{b\to\infty} \int_0^b \frac{x}{(1+x)^3} dx$$

$u = x \quad v' = (1+x)^{-3}$

$u' = 1 \quad v = \dfrac{-1}{2(1+x)^2}$

$$= c \lim_{b\to\infty} \frac{-x}{2(1+x)^2} \Big|_0^b + \int_0^b \frac{1}{2(1+x)^2} dx$$
$$= c \lim_{b\to\infty} \frac{-x}{2(1+x)^2} - \frac{1}{2(1+x)} \Big|_0^b = \frac{c}{2} = 1.$$

or

$$\text{mean} = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} c \frac{x}{(1+x)^3} dx = c \lim_{b\to\infty} \int_0^b \frac{x}{(1+x)^3} dx$$

$u = 1 + x$

$$= c \lim_{b\to\infty} \int_1^{b+1} \frac{u-1}{u^3} dx = c \lim_{b\to\infty} \int_1^{b+1} u^{-2} - u^{-3} dy$$

$$= c \lim_{b\to\infty} -u^{-1} + \frac{u^{-2}}{2} \Big|_1^{b+1} = \frac{c}{2} = 1.$$

d. [1 point] A second machine produces the same type of wire, but with a different probability density function (pdf). Which of the following graphs could be the graph of the pdf for the second machine? Circle all your answers.

Solution: The graph on the left upper corner can't be the density since$x$ is the distance between flaws, hence the probability density function can't be positive for $x < 0$.
The graph on the left lower corner can't be the density since the area under the curve for $x \geq 0$ is infinite (it has a positive horizontal asymptote).
The graph on the right upper corner can't be a density since it is negative on an interval.