Math 116 — Final Exam

April 19, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 16 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	13	
3	12	
4	11	
5	8	
6	9	
7	8	
8	8	
9	10	
10	9	
Total	100	

You may find the following expressions useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x < 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \cdots \qquad \text{for } -1 < x < 1$$

1. [12 points] Indicate whether each of the following statements are true or false by circling the correct answer. **You do not need to justify your answers.**

a. [2 points] The curve defined by the parametric equations $x = 1 - \cos t$ and $y = t - \sin t$ has a vertical tangent line when $t = \pi$.

$\boxed{\text{True}}$ False

Solution: $x'(\pi) = \sin \pi = 0$ and $y'(\pi) = 1 - \cos(\pi) = 2$, hence the curve has a vertical tangent at $t = \pi$.

b. [2 points] If the sequence a_n converges to 0 and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

True $\boxed{\text{False}}$

Solution: Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, then $\frac{1}{n}$ converges to 0 and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty}(\frac{1}{n} + \frac{1}{n^2}) \geq \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

c. [2 points] The graph of a polar function $r = f(\theta)$ in the (x, y)-plane has a horizontal tangent line at $\theta = a$ if $f'(a) = 0$.

True $\boxed{\text{False}}$

Solution: The graph of $r = f(\theta) = \cos \theta$ satisfies $f'(\pi) = -\sin \pi = 0$, but it has no horizontal asymptote at $\theta = \pi$.

d. [2 points] The integral $\int_0^1 \pi x^4 dx$ computes the volume of the solid obtained by rotating the graph of $y = x^2$ around the x axis for $0 \leq x \leq 1$.

$\boxed{\text{True}}$ False

Solution: $V = \int_0^1 \pi(x^2)^2 dx = \int_0^1 \pi x^4 dx$

e. [2 points] Let $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} x^n$ be the Taylor series of $f(x)$ about 0. Then $f(x)$ is concave up at $x = 0$.

$\boxed{\text{True}}$ False

Solution: The coefficient of x^2 in the Taylor series is $\frac{f''(0)}{2!} = \frac{1}{2^2+1} = \frac{1}{5} > 0$. Hence $f''(0) > 0$ so $f(x)$ is concave up.

f. [2 points] The integral test says that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \int_1^{\infty} \frac{1}{x^2} dx$.

True $\boxed{\text{False}}$

Solution: The integral test only says $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ behaves as $\displaystyle\int_1^{\infty} \frac{1}{x^2}dx$.

2. [13 points] Determine if each of the following sequences is increasing, decreasing or neither, and whether it converges or diverges. Circle all the answers that apply. On parts **a-c**, if the sequence converges, find the limit. No justification is required.

a. [3 points] For $n \geq 1$, let $a_n = 3 + \frac{1}{n}$.

Solution:

1. Increasing $\boxed{\textbf{Decreasing}}$ Neither.

2. $\boxed{\textbf{Convergent}}$: $\lim\limits_{n\to\infty} a_n = 3$ Divergent

b. [3 points] For $n \geq 1$, let $a_n = (-\frac{\pi}{e})^n$.

Solution:

1. Increasing Decreasing $\boxed{\textbf{Neither.}}$

2. Convergent: $\lim\limits_{n\to\infty} a_n =$ __________ $\boxed{\textbf{Divergent}}$

c. [3 points] Let $P(x)$ be the cumulative distribution function of a nonzero probability density function $p(x)$. Define $a_n = P(n)$ for $n \geq 1$.

Solution:

1. $\boxed{\textbf{Increasing}}$ Decreasing Neither.

2. $\boxed{\textbf{Convergent}}$: $\lim\limits_{n\to\infty} a_n = 1$ Divergent

d. [2 points] For $n \geq 1$, let $a_n = 1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \cdots + \dfrac{(-1)^n}{n!}$.

Solution:

1. Increasing Decreasing $\boxed{\textbf{Neither}}$.

2. $\boxed{\textbf{Convergent}}$ (no need to compute the limit) Divergent

e. [2 points] Let $a_n = \displaystyle\int_2^n \frac{1}{\sqrt{x}-1}\,dx$, for $n \geq 2$.

Solution:

1. $\boxed{\textbf{Increasing}}$ Decreasing Neither.

2. Convergent (no need to compute the limit) $\boxed{\textbf{Divergent}}$

3. [12 points]

a. [6 points] State whether each of the following series converges or diverges. Indicate which test you use to decide. Show all of your work to receive full credit.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: The function $\frac{1}{n\sqrt{\ln n}}$ is decreasing and positive for $n \geq 2$, then the Integral test says that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ behaves as $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} u^{-\frac{1}{2}}du = \lim_{b\to\infty} 2\sqrt{u}\,\Big|_{\ln 2}^{\ln b} = \infty.$$

Hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$

Solution: Since $0 \leq \dfrac{\cos^2(n)}{\sqrt{n^3}} \leq \dfrac{1}{n^{\frac{3}{2}}}$, and $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by p-series test ($p = \frac{3}{2} > 1$), then comparison test yields the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{\sqrt{n^3}}$.

b. [6 points] Decide whether each of the following series converges absolutely, converges conditionally or diverges. Circle your answer. No justification required.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$

Converges absolutely **Converges conditionally** Diverges

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{3n}}{5^n}$

Converges absolutely Converges conditionally **Diverges**

Solution: $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right| = \sum_{n=0}^{\infty}\frac{\sqrt{n^2+1}}{n^2+n+8}$ behaves as $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ since

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n^2+1}}{n^2+n+8}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n\sqrt{n^2+1}}{n^2+n+8} = 1 > 0.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (p-series test $p=1$), then by limit comparison test $\displaystyle\sum_{n=0}^{\infty}\left|\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}\right|$ diverges.

The convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{n^2+1}}{n^2+n+8}$ follows from alternating series test since for $a_n = \frac{\sqrt{n^2+1}}{n^2+n+8}$:

- $\lim_{n\to\infty} a_n = 0$.
- a_n is decreasing

$$\frac{d}{dn}\left(\frac{\sqrt{n^2+1}}{n^2+n+8}\right) = \frac{-1+6n-n^3}{\sqrt{1+n^2}(n^2+n+8)^2} < 0$$

for n large.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^{3n}}{5^n} = \sum_{n=0}^{\infty}\left(-\frac{8}{5}\right)^n$ is a geometric series with ratio $r = -\frac{8}{5} < -1$, hence it diverges.

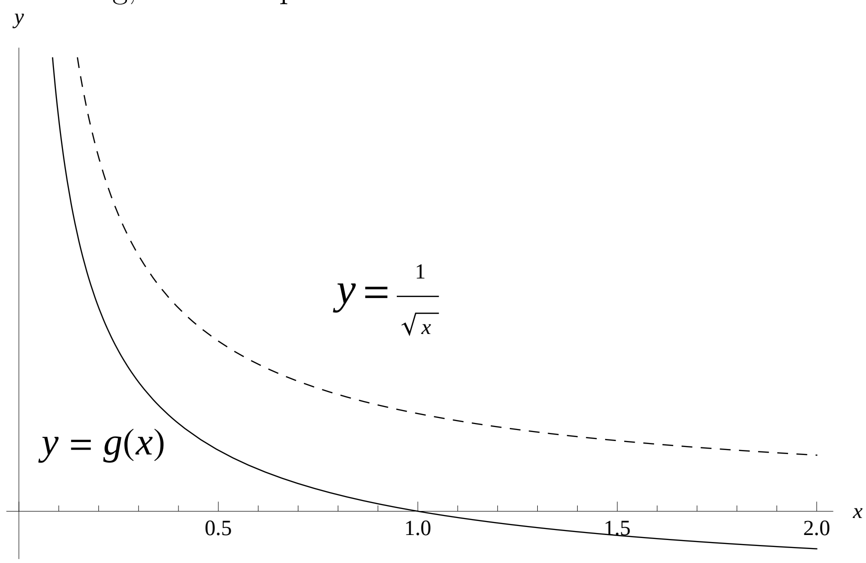
4. [11 points]

a. [2 points] Let $g(x)$ be a continuous function for $x > 0$ and let $G(x)$ be the antiderivative of $g(x)$ with $G(1) = 0$. Write a formula for $G(x)$.

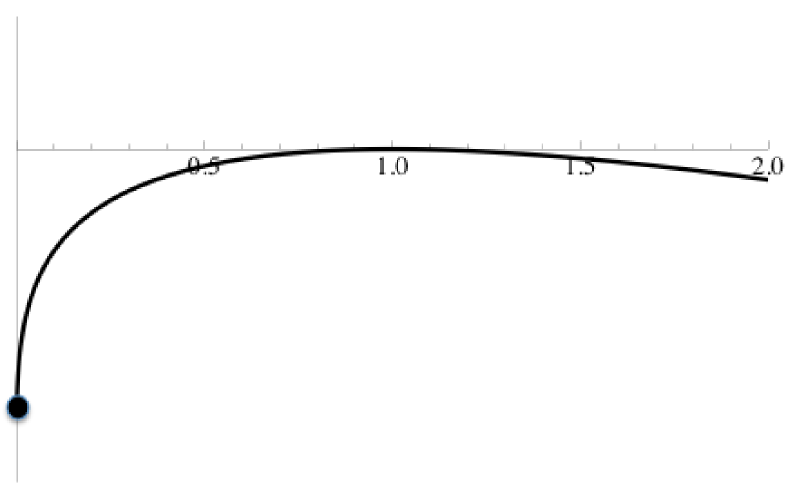
Solution: $G(x) = \int_1^x g(t)dt$

b. [5 points] The graph of $g(x)$ is shown below. The function $g(x)$ has a vertical asymptote at $x = 0$ and $g(x) < \dfrac{1}{\sqrt{x}}$ for $x > 0$.

Sketch the graph of $G(x)$ for $0 \le x \le 2$. Make sure you indicate where $G(x)$ has asymptotes, local maxima, or local minima, as well as where $G(x)$ is increasing, decreasing, concave up or concave down.



Solution:



c. [4 points] Suppose $h(x)$ and $f(x)$ are continuous functions satisfying

i. $0 < f(x) \leq \dfrac{1}{x^p}$ for $0 < x \leq 1$.

ii. $\dfrac{1}{x^{p+\frac{1}{2}}} \leq h(x) \leq \dfrac{1}{x^p}$ for $x \geq 1$.

Decide whether each of the following expressions converge, diverge or if there is not enough information available to conclude.

Solution:

i. If $p = \dfrac{1}{2}$,

(a) $\lim\limits_{x\to\infty} h(x)$

$\boxed{\textbf{Converges}}$ Diverges Not possible to conclude.

(b) $\displaystyle\int_1^\infty h(x)dx$:

Converges $\boxed{\textbf{Diverges}}$ Not possible to conclude.

ii. If $p = 2$,

(a) $\displaystyle\int_1^\infty h(x)dx$:

$\boxed{\textbf{Converges}}$ Diverges Not possible to conclude.

(b) $\displaystyle\int_0^1 f(x)dx$

Converges Diverges $\boxed{\textbf{Not possible to conclude.}}$

5. [8 points] Consider

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n}.$$

a. [2 points] Does the series converge for $x = 2$? Justify your answer.

Solution: At $x = 2$

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

The series diverge since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

b. [2 points] Based only on your answer from part **a**, what can you say about R, the radius of convergence of the series? Circle your answer.

Solution: $R = 2$ $\quad R > 2$ $\quad R < 2$ $\quad \boxed{R \leq 2}$ $\quad R \geq 2$

since it is possible for $x = 2$ to be one of the endpoints in the interval of convergence.

c. [4 points] Find the interval of convergence of the series.

Solution:

$$\lim_{n\to\infty} \frac{\left|\frac{n+1}{4^{n+1}(n+2)}\, x^{2n+2}\right|}{\left|\frac{n}{4^n(n+1)}\, x^{2n}\right|} = x^2 \lim_{n\to\infty} \frac{(n+1)^2}{4n(n+2)} = \frac{x^2}{4}.$$

If $\frac{x^2}{4} < 1$, then the series converges. Hence Ratio test states that the series converges if $-2 < x < 2$. We need to check the endpoints $x = \pm 2$. We already checked $x = 2$. For $x = -2$,

$$\sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, x^{2n} = \sum_{n=1}^{\infty} \frac{n}{4^n(n+1)}\, 4^n = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges by part **a**. The interval of convergence of the series is $-2 < x < 2$.

6. [9 points] Let $y(t)$ be the number of fish (**in hundreds**) in an artificial lagoon, where t is measured in years. The function $y(t)$ satisfies the following differential equation

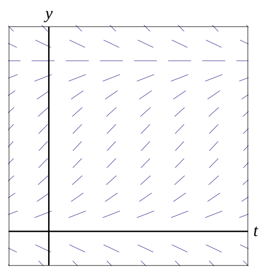
$$\frac{dy}{dt} = y\,(10 - y) - h.$$

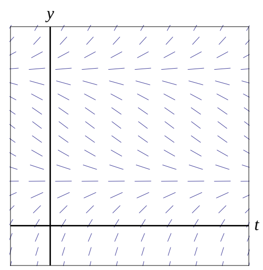
where the constant h is the rate at which the fish are harvested from the lagoon.

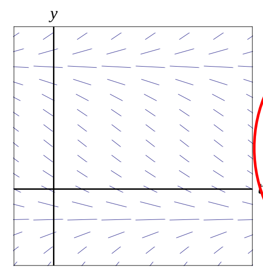
a. [4 points] Suppose there is no harvesting ($h = 0$). Find the equilibrium solutions of the differential equation. Determine the stability of each equilibrium.

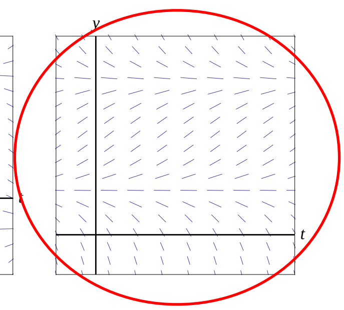
Solution: Equilibrium solutions are found by setting $\frac{dy}{dt} = 0$. So when $h = 0$, we have $y = 0$ and $y = 10$ as equilibrium solutions. Now when $y < 0$ or $y > 10$, we have $\frac{dy}{dt} < 0$. For $0 < y < 10$, we have $\frac{dy}{dt} > 0$. So $y = 0$ is an unstable equilibrium solution and $y = 10$ is a stable equilibrium solutions.

b. [2 points] Suppose the fish are harvested at a rate $h = 9$. Which of the following slope fields may correspond to the differential equation for $y(t)$? Circle your answer.







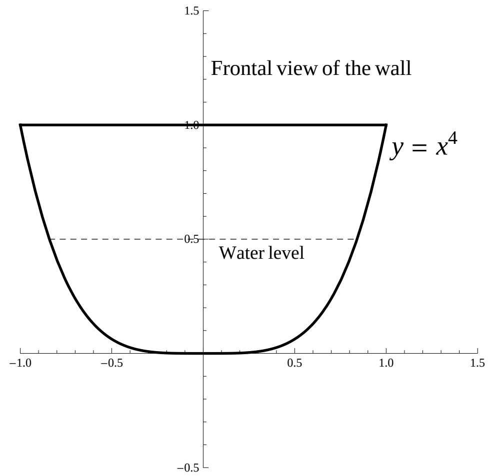


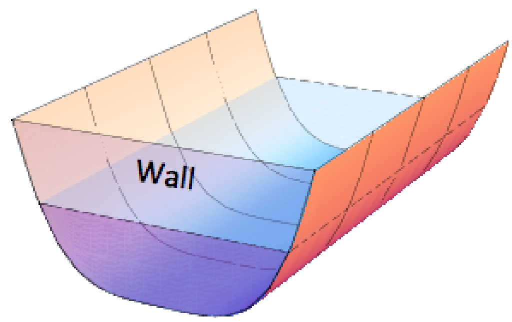
Solution: The equation is $y' = y(10 - y) - 9$. The equilibrium solutions are $y = 1$ (unstable) and $y = 9$ (stable).

c. [3 points] If (at $t = 0$) there are 200 fish in the lagoon, what is the maximum rate h for harvesting the fish, while still maintaining the fish population in the long run (i.e do not let the fish die out)? (Hint: You do not need to solve the differential equation to answer this question).

Solution: In order for the fish to not die out, we need $\frac{dy}{dt} \geq 0$. That gives us the equation $16 - h \geq 0$, so $h \leq 16$. Therefore 16 is the maximum rate of harvesting for the fish.

7. [8 points] A canal with cross sectional area given by the graph of the function $y = x^4$ (where x and y are given in meters) holds water to the depth of $\frac{1}{2}$ meter as shown in the figure below. The water is contained in the canal by a wall which is perpendicular to the canal. The density of water is $1,000$ kg/m^3. Be sure to include units.





a. [6 points] Write an expression that approximates the force of the water on a horizontal slice of the wall that is y meters above the bottom of the canal and has thickness Δy.

Solution:

$$\text{Pressure} = 1000(9.8)\left(\frac{1}{2} - y\right) \qquad \text{Area} = 2\sqrt[4]{y}\Delta y$$

$$\text{Force}_{\text{slice}} = (\text{Pressure})\ (\text{Area}) = 1000(9.8)\left(\frac{1}{2} - y\right)(2\sqrt[4]{y})\Delta y \quad \text{Newtons}$$

b. [2 points] Find an expression involving definite integrals that represents the total force of the water on the wall. You do not need to evaluate the integrals.

Solution:

$$F = \int_0^{\frac{1}{2}} 1000(9.8)\left(\frac{1}{2} - y\right)(2\sqrt[4]{y})dy \quad \text{Newtons}$$

8. [8 points] The function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

computes the arc length of the graph of the function $y = t^3$ from $t = 0$ to $t = x$.

a. [4 points] Approximate the value of $F(\frac{1}{2})$, the arc length of the curve $y = t^3$ for $0 \le t \le \frac{1}{2}$, using $RIGHT(2)$, $LEFT(2)$, $TRAP(2)$ and $MID(2)$. Write each term of each sum to receive full credit.

Solution: We have $\Delta x = 1/4$. Using this we have

$$LEFT(2) = \tfrac{1}{4}\left(1 + \sqrt{1+9\left(\tfrac{1}{4}\right)^4}\right) \approx 0.504.$$

$$RIGHT(2) = \tfrac{1}{4}\left(\sqrt{1+9\left(\tfrac{1}{4}\right)^4} + \sqrt{1+9\left(\tfrac{1}{2}\right)^4}\right) \approx 0.566.$$

$$TRAP(2) = \tfrac{1}{2}(LEFT(2) + RIGHT(2)) \approx 0.535.$$

$$MID(2) = \tfrac{1}{4}\left(\sqrt{1+9\left(\tfrac{1}{8}\right)^4} + \sqrt{1+9\left(\tfrac{3}{8}\right)^4}\right) \approx 0.521.$$

b. [2 points] Which approximation $RIGHT$ or $LEFT$ is guaranteed to give an underestimate for $F(\frac{1}{2})$? Justify.

Solution: The integrand $f(x) = \sqrt{1+9x^4}$ is an increasing function, so $LEFT(2)$ is an underestimate.

c. [2 points] Find $F'(1)$.

Solution: Using the second fundamental theorem of calculus, we have

$$F'(x) = \sqrt{1+9x^4}$$

so $F'(1) = \sqrt{10}$.

9. [10 points] A second way to approximate the function

$$F(x) = \int_0^x \sqrt{1+9t^4}dt.$$

is by using its Taylor polynomials.

a. [2 points] Find the first three nonzero terms in the Taylor series for the function $\sqrt{1+u}$ about $u = 0$.

Solution:

$$\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$$

b. [2 points] Find the first three nonzero terms in the Taylor series for $\sqrt{1+9t^4}$ about $t = 0$.

Solution: Let $u = 9t^4$ then

$$\sqrt{1+9t^4} \approx 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8$$

c. [2 points] Find the first three nonzero terms in the Taylor series for $F(x)$ about $x = 0$.

Solution:

$$F(x) \approx \int_0^x 1 + \frac{9}{2}t^4 - \frac{81}{8}t^8 dt = x + \frac{9}{10}x^5 - \frac{9}{8}x^9$$

d. [2 points] For which values of x do you expect the Taylor series for $F(x)$ about $x = 0$ to converge? Justify your answer.

Solution: We substituted $u = 9t^4$ into the Binomial series. The interval of convergence for the Binomial series is $-1 < u < 1$. Then we expect the series to converge for $0 \leq 9x^4 < 1$. Hence the Taylor series for $F(x)$ about $x = 0$ converges if $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

e. [2 points] Use the fifth degree Taylor polynomial for $F(x)$ about $x = 0$ to approximate the value of $F(\frac{1}{2})$.

Solution: $P_5(x) = x + \frac{9}{10}x^5$, then $F\left(\frac{1}{2}\right) \approx P_5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{9}{10}\left(\frac{1}{2}\right)^5 = 0.528$

10. [9 points] A patient takes a drug in doses of 100 mg once every 24 hours. The half-life of the drug in the patient's body is 12 hours. Let D_n be the amount of the drug in the patient immediately after taking the nth dose of the drug. Be sure to include units.

a. [3 points] Find D_1, D_2 and D_3.

Solution: Since the half-life is 12 hours, after 24 hours, $\frac{1}{4}$ of the drug remains in the body.

$$\begin{aligned} D_1 &= 100\text{mg} \\ D_2 &= 100 + 100\left(\tfrac{1}{4}\right) = 100\left(1 + \tfrac{1}{4}\right) = 125\text{mg} \\ D_3 &= 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 100\left(1 + \tfrac{1}{4} + \tfrac{1}{16}\right) = 131.25 \text{ mg} \end{aligned}$$

b. [4 points] Find a closed form expression (an expression that does not involve a long summation or a recursive formula) for D_n.

Solution: From part **a**

$$D_n = 100 + 100\left(\tfrac{1}{4}\right) + 100\left(\tfrac{1}{4}\right)^2 + \cdots + 100\left(\tfrac{1}{4}\right)^{n-1}.$$

This is a finite geometric series with first term 100 and the common ratio between terms is $\frac{1}{4}$. So we have

$$D_n = \sum_{k=1}^{n} 100\left(\tfrac{1}{4}\right)^{k-1} = \frac{100\left(1 - \left(\frac{1}{4}\right)^n\right)}{\frac{3}{4}} = \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) \text{ mg}$$

c. [2 points] What is $\lim_{n\to\infty} D_n$?

Solution: Taking the limit, we obtain

$$\lim_{n\to\infty} \frac{400}{3}\left(1 - \left(\tfrac{1}{4}\right)^n\right) = \frac{400}{3} \approx 133.3 \text{ mg}$$