Math 116 — First Midterm

## February 11, 2013

Name: _____ EXAM SOLUTIONS _____

Instructor: _____ Section: _____

1. **Do not open this exam until you are told to do so.**

2. This exam has 14 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.

5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.

6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.

7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.

8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	14	
3	14	
4	8	
5	15	
6	11	
7	8	
8	11	
9	9	
Total	100	

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Let $(\bar{x}, \bar{y})$ be the center of mass of the metal plate bounded by the line $y = 1 - x$, the x-axis and the y-axis for $0 \le x \le 1$.
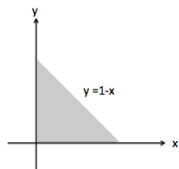
If the plate has uniform mass density, then $\bar{x} = \bar{y}$.

|True| False

> *Solution:* Since the plate has uniform density and it is symmetric about the line $y = x$, then $\bar{y} = \bar{x}$. It also follows using the formulas for the coordinates of the center of mass
> $$\bar{x} = \frac{\int_0^1 \delta x(1-x)\,dx}{\int_0^1 \delta(1-x)\,dx} = \frac{\int_0^1 \delta y(1-y)\,dy}{\int_0^1 \delta(1-y)\,dy} = \bar{y}.$$

b. [2 points] The function $F(x) = \displaystyle\int_1^{x^2} \sin(e^t)\,dt$ is an even function.

|True| False

> *Solution:* The function $F(x)$ is even if it satisfies $F(-x) = F(x)$. Since
> $$F(-x) = \int_1^{(-x)^2} \sin(e^t)\,dt = \int_1^{x^2} \sin(e^t)\,dt = F(x).$$
> then $F(x)$ is even.

c. [2 points] Let $h(x)$ be an antiderivative of $g(x)$. If $g(x)$ is measured in kg and x in inches, then the units for $h(x)$ are kg per inch.

True |False|

> *Solution:*   The second fundamental theorem of calculus says that $h(x) = \displaystyle\int_a^x g(t)\,dt$ for some contant a. The units of $g(x)$ and x are kg and inches respectively, then the units of $h(x)$ are kg $\cdot$ inches.

d. [2 points] The function $R(t) = \displaystyle\int_t^{1-t} e^{x^3}\,dx$ is decreasing for all values of t.

|True| False

> *Solution:*
> $$R'(t) = -e^{(1-t)^3} - e^{t^3} < 0 \text{ for all values of } t.$$
> Hence $R(t)$ is decreasing.

e. [2 points] The length of the curve $y = x^2$ from $x = 0$ to $x = 2$ is smaller than 4.

<div align="right">True   <u>False</u></div>

Solution: The length L of the curve $y = x^2$ from $x = 0$ to $x = 2$ is given by

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{1 + (2x)^2}\, dx.$$

Using your calculator you can find $L \approx 4.64$. Or, you can argue without using your calculator. The length L of the curve $y = x^2$ from $x = 0$ to $x = 2$ is larger than the length L_1 of the line connecting the points $(0,0)$ and $(2,4)$. Since $L_1 = \sqrt{(2-0)^2 + (4-0)^2} > \sqrt{4^2} = 4$. Hence $L > L_1 > 4$.

2. [14 points] Let $f(x)$ be a continuous function on $0 \leq x \leq 2$. The values of $f(x)$ are shown below

x	0	0.5	1	1.5	2
$f(x)$	-3	-2	1	3	4

a. [2 points] Use the left-hand sum with four subintervals to approximate the value of $\int_0^2 f(x)dx$. Show all the terms in the sum, and then calculate the numerical value.

Solution:
$$\text{Left}(4) = \frac{1}{2}(-3 - 2 + 1 + 3) = -\frac{1}{2}.$$

b. [2 points] Assume that $f(x)$ has no critical points for $0 \leq x \leq 2$. Is your estimate in (**a**) guaranteed to be an underestimate or overestimate of $\int_0^2 f(x)dx$, or there is not enough information to decide? Justify.

Solution: Underestimate, because $f(x)$ is increasing.

c. [2 points] Use the trapezoid rule with four subintervals to approximate the value of $\int_0^2 f(x)dx$. Show all the terms in the sum, and then calculate the numerical value.
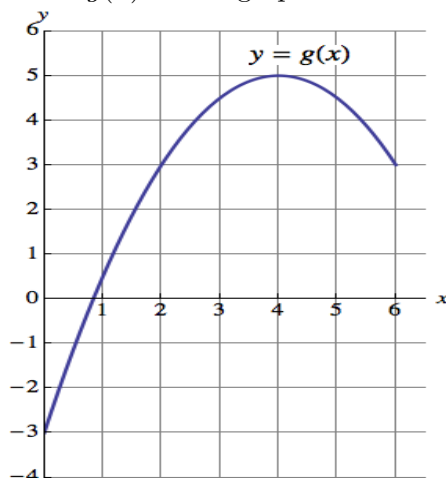
Solution:
$$\frac{\frac{1}{2}(-3 - 2 + 1 + 3) + \frac{1}{2}(-2 + 1 + 3 + 4)}{2} = \frac{-\frac{1}{2} + 3}{2} = 1.25.$$

d. [2 points] Given the data for $f(x)$, is your estimate in (**c**) guaranteed to be an underestimate or overestimate of $\int_0^2 f(x)dx$, or there is not enough information to decide? Justify.

Solution: There's not enough information to decide, because the data shows that f is not always concave up or always concave down.

e. [2 points] Consider the function $g(x)$ whose graph is shown below

Use the midpoint rule with three subintervals to approximate the value of $\int_0^6 g(x)dx$. Show all the terms in the sum, and then calculate the numerical value.

> *Solution:* $\text{Mid}(3) = 2(.5 + 4.5 + 4.5) = 19$.

f. [2 points] Use the right-hand sum with three subintervals to approximate the value of $\int_1^3 e^{\sqrt{t}}dt$. Show all the terms in the sum, and then calculate the numerical value.

> *Solution:*
> $$\text{Right}(3) = \frac{2}{3}(e^{\sqrt{5/3}} + e^{\sqrt{7/3}} + e^{\sqrt{3}}) \approx \frac{2}{3}(3.636 + 4.606 + 5.652) \approx 9.262$$

g. [2 points] Is your estimate in (**f**) guaranteed to be an underestimate or overestimate of $\int_1^3 e^{\sqrt{t}}dt$, or there is not enough information to decide? Justify.

> *Solution:*   We calculate
> $$\frac{d}{dt}e^{\sqrt{t}} = \frac{1}{2}\left(\frac{1}{\sqrt{t}}e^{\sqrt{t}}\right) > 0,$$
> so $e^{\sqrt{t}}$ is increasing. Therefore $\text{Right}(3)$ is an overestimate.

3. [14 points] Consider functions $f(x)$ and $g(x)$ satisfying:

    (i) $g(x)$ is an odd function.

    (ii) $\displaystyle\int_2^7 g(x)dx = 3.$

    (iii) $\displaystyle\int_2^7 f(x)dx = 17.$

    (iv) $f(2) = 1.$

    (v) $\displaystyle\int_1^6 f'(x)dx = 12.$

    (vi) $\displaystyle\int_2^7 f'(x)dx = 3.$

Compute the value of the following quantities. If it is impossible to determine their value with the information provided above, write "NI" (not enough information).

a. [2 points] $\displaystyle\int_{-2}^7 g(x)\ dx = $ _____

> *Solution:* 3, using i and vi.

b. [2 points] $\displaystyle\int_2^7 (f(x) - 8g(x))\ dx = $ _____

> *Solution:* -7, using ii and iii.

c. [2 points] $f(7) = $ _____

> *Solution:* 4, using the Fundamental Theorem of Calculus with iv and vi.

d. [2 points] $\displaystyle\int_1^6 f'(x+1)\ dx = $ _____

> *Solution:* We use u substitution, $u = x + 1$. Making sure to change the limits of integration, we get $\displaystyle\int_2^7 f'(u)du = 3.$

**e.** [3 points] $\displaystyle\int_2^7 xf'(x)\ dx = $ _____

> *Solution:* We integrate by parts with $u = x, dv = f'$.
>
> $$\int_2^7 xf'(x)dx = xf(x)|_2^7 - \int_2^7 f(x)dx = (7f(7) - 2f(2)) - 17 = 28 - 2 - 17 = 9.$$

f. [3 points] $\displaystyle\int_2^3 xf(x^2 - 2)\ dx = $ _____

> *Solution:* We use u substitution $u = x^2 - 2, du = 2xdx$. We get $\dfrac{1}{2}\displaystyle\int_2^7 f(u)du = \dfrac{17}{2} = 8.5.$

4. [8 points] In a small town, property values close to the school are determined primarily by how far the land is from the school. The function $\delta(r) = \dfrac{1}{ar^2 + 1}$ gives the value of the land (in thousands of dollars per m^2), where r is the distance (in meters) from the school and a is a positive contant.

a. [5 points] Find a formula containing a definite integral that computes the value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m (figure shown below).
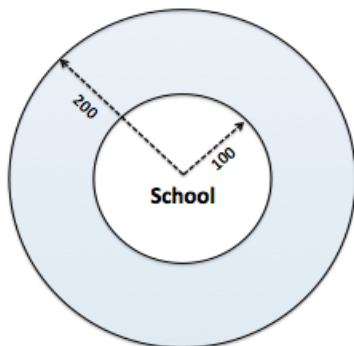
Solution: A thin annular slice has area $A_{slice} \approx 2\pi r \Delta r$, and so has an approximate value $V_{slice} \approx \frac{1}{ar^2+1} 2\pi r \Delta r$. Summing these slice up and taking the limit as $\Delta r \to 0$, we get the integral

$$\text{Value of the land in the annulus} = \lim_{\Delta r \to 0} \sum \frac{1}{ar^2 + 1} 2\pi r \Delta r = \int_{100}^{200} 2\pi r \frac{1}{ar^2 + 1} dr.$$

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b. [3 points] Calculate the exact value of the land that lies in the annulus of inner radius 100 m and outer radius 200 m. Your answer should contain a. Show all your work.

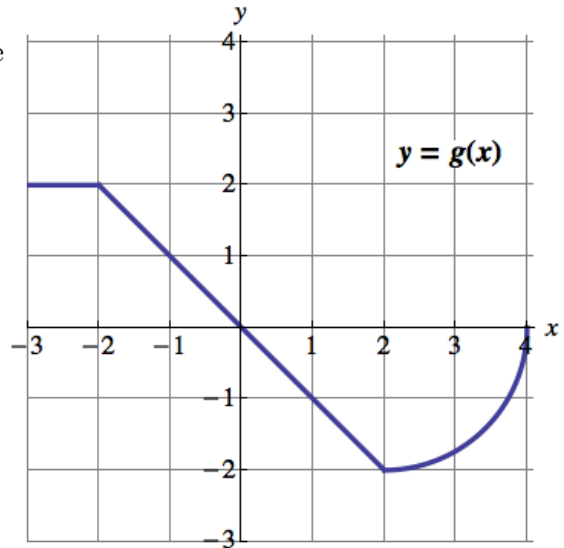*Solution:*   Let $u = ar^2 + 1$. Then $du = 2ar\, dr$. Our integral becomes

$$\int_{a100^2+1}^{a200^2+1} 2\pi r \frac{1}{u} \frac{du}{2ar} = \frac{\pi}{a} \int_{10,000a+1}^{40,000a+1} \frac{du}{u} = \frac{\pi}{a} \left(\ln(40,000a + 1) - \ln(10,000a + 1) \right).$$

5. [15 points]

a. [11 points] Let $G(x) = \int_{-2}^{x} g(t)\,dt$ where the graph of the function $g(x)$ is shown below.
The graph of $g(x)$ is a quarter of a circle for $2 \le x \le 4$.

Fill in the indicated values of $G(x)$ in the table below.

x	-3	-2	0	2	4
$G(x)$					

$y = g(x)$

Draw the graph of $G(x)$ for $-3 \le x \le 4$. Make sure your graph indicates the regions where the function $G(x)$ is increasing, decreasing, concave up or concave down, and appropriately reflects the critical points of $G(x)$.

$y = G(x)$
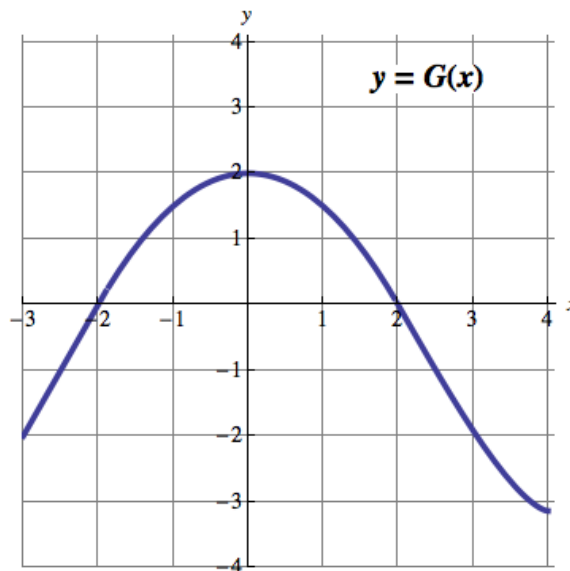
Solution:

x	-3	-2	0	2	4
$G(x)$	-2	0	2	0	$-\pi$

b. [4 points] Consider the function

$$f(x) = \begin{cases} -x & \text{for } x \le 0 \\ x^2 & \text{for } 0 < x. \end{cases}$$

Let $F(x)$ be an antiderivative of $f(x)$ with $F(-2) = 0$. Find a formula for $F(x)$. Your answer should not include any integrals.

$$F(x) = \begin{cases} \underline{\hspace{4cm}} & \text{for } x \le 0 \\ \\ \underline{\hspace{4cm}} & \text{for } 0 < x. \end{cases}$$
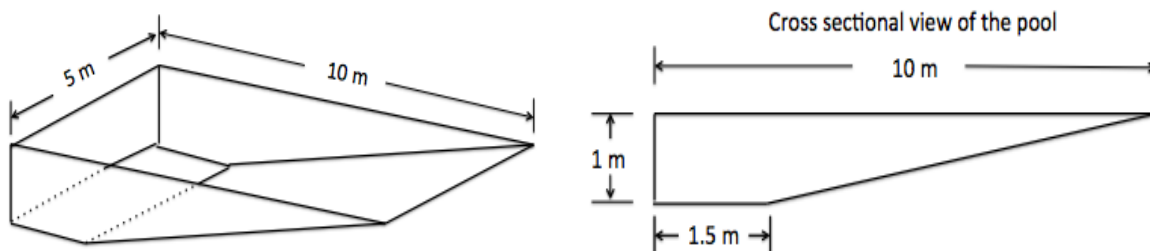
Solution: We know that $F(x) = \int_{-2}^{x} f(t)dt$. For $x \le 0$, this tells us that $F(x) = \int_{-2}^{x}(-t)dt = \frac{-x^2}{2} + 2^2/2 = -x^2/2 + 2$.
For $x > 0$, we have

$$\int_{-2}^{x} f(t)dt = \int_{-2}^{0}(-t)dt + \int_{0}^{x} t^2 dt = 2 + x^3/3.$$

$$F(x) = \begin{cases} -x^2/2 + 2 & \text{for } x \le 0 \\ \\ 2 + x^3/3 & \text{for } 0 < x. \end{cases}$$

6. [11 points] A swimming pool 10 m long and 5 m wide has varying depth. Its maximum depth is 1 m as shown in the picture below

Cross sectional view of the pool

The swimming pool has water up to a level of maximum depth of 0.6 m. The density of water is 1000 kg per m³. Use $g = 9.8$ m/s² for the acceleration due to gravity.

**a.** [9 points] Write an expression that approximates the work done in lifting a horizontal slice of water with thickness Δy meters, that is at a distance of y meters above the bottom, to the top of the swimming pool.

Solution: First we must find a formula for the length of the swimming pool at depth for a given height above the bottom. Let's call this function $l(y)$. We know that $l(0) = 1.5$ and $l(1) = 10$. Since $l(y)$ is a linear function, this tells us that $l(y) = 8.5y + 1.5$. The volume of such a slice is $\Delta y(8.5y + 1.5) \cdot 5$. Multiplying by 1000 kg/m³ and 9.8 m/s² gives us the weight of the water in Newtowns. The amount the water needs to be lifted is $(1 - y)$. We therefore get:

$$W_{slice} \approx 1000 \cdot 9.8 \cdot (8.5y + 1.5) \cdot 5 \cdot (1 - y)\Delta y.$$

b. [2 points] Write a definite integral that computes the work required to pump all the water to the top of the pool.

Cross sectional view of the pool
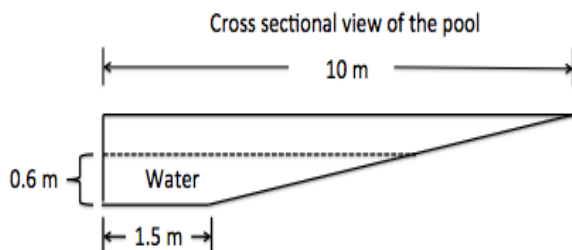
Solution: Work $= \displaystyle\int_0^{0.6} 1000 \cdot 9.8 \cdot 5(8.5y + 1.5)(1 - y)dy$ Joules.

7. [8 points] Let S be the solid whose base is the region bounded by the curves $y = x^2$, $y = 6 - x$ and $x = 0$ and whose cross sections **parallel** to the x-axis are squares. Find a formula involving definite integrals that computes the volume of S.
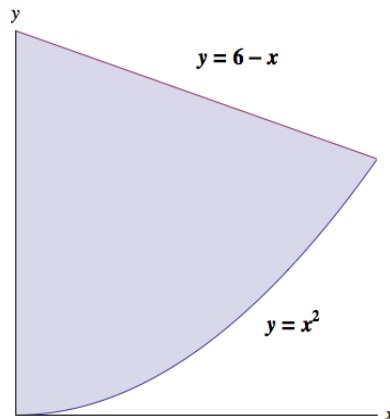

Solution: First we solve for where the two curves intersect. $6 - x = x^2$ implies $0 = x^2 + x - 6 = (x+3)(x-2)$, so $x = 2$, which implies $y = 4$. We have to split the problem into two cases, one when $0 \leq y \leq 4$, and one when $4 \leq y \leq 6$. We will choose thin horizontal slices, and integrate in terms of y, so we need to solve for x in terms of y: $x = \sqrt{y}$ and $x = 6 - y$ are our two curves.

In the case of $0 \leq y \leq 4$, a thin slice has volume $V_{slice} \approx (\sqrt{y})^2 \Delta y$. In the second case, $4 \leq y \leq 6$, a thin slice has volume $V_{slice} \approx (6 - y)^2 \Delta y$. Hence the total volume of the solid is given by

$$V = \int_0^4 y\,dy + \int_4^6 (6 - y)^2\,dy.$$

8. [11 points] A tortoise and a hare decide to race. They decide to race a straight 5 kilometer course. The race starts at 12pm. The hare is much faster than the tortoise, so he's confident that he'll win. The hare runs very fast for 30 minutes, getting to what it knows is the half-way point. The hare is tired (it had been studying for exams the night before), so it decides to take a nap. It falls asleep for 5 hours, wakes up, discovers that (now that it's 5:30) it's dark, and runs to the finish line, arriving at 6pm. When it gets there, it's surprised to see the tortoise is already there. "I hope you enjoyed your nap! I've been here for an hour, since 5 o'clock!" the tortoise says. "Steady and slow is the way to go: I kept going the same speed the whole time."

Let $H(t)$ be the hare's velocity and $T(t)$ be the tortoise's velocity, in km per hour, where t is measured in hours after 12pm.
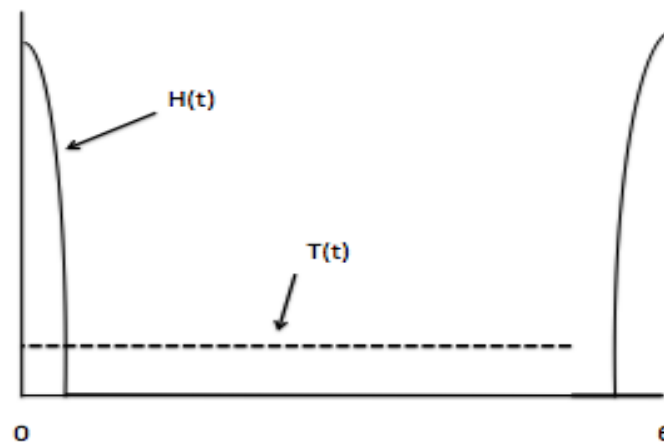
Let

$$R(t) = \int_0^t H(s)ds - \int_0^t T(s)ds.$$

a. [1 point] At times when $R(t) > 0$, who is winning the race?

Solution: The hare

b. [2 points] What is the practical interpretation of the function $|R(t)|$? Include units.

Solution: $|R(t)|$ is the distance in km between the tortoise and the hare t hours after 12pm.

c. [3 points] For what values of $0 \leq t \leq 6$, does $R(t) = 0$?

*Solution:* $t = 0, t = 2.5, t = 6$.

d. [2 points] For what values of $0 \leq t \leq 6$ is the function $\dfrac{dR}{dt} < 0$?

Solution: $0.5 < t < 5$.

e. [3 points] Write down a definite integral that represents the hare's average velocity from 12 to 12:30. What is the value of the hare's average velocity during this time?

Solution: $\frac{1}{.5} \int_0^{1/2} H(s)ds$. We know that $\int_0^{1/2} H(s)ds = 2.5$, because the Hare has gotten halfway by 12:30. Therefore, the average velocity is 5 km/hr.

9. [9 points] Consider the region R bounded by the curves $y = x^2$, $y = x + 2$ and the y-axis, where x and y are measured in meters.
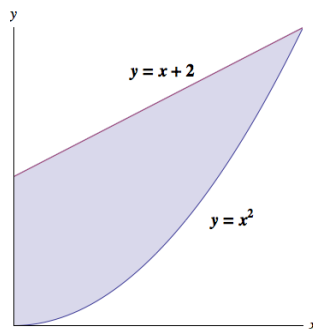
a. [5 points] Let T be the solid obtained by rotating the region R about the x-axis. Find a formula involving definite integrals that computes the volume of T.

Solution: Using washers: $V = \displaystyle\int_0^2 \pi[(x+2)^2 - x^4]dx$.

Using shells: $V = \displaystyle\int_0^2 2\pi y\sqrt{y}dy + \int_2^4 2\pi y(\sqrt{y} - (y-2))dy$

b. [2 points] The mass density of the solid T is given by the function $\delta(x) = 2 - \sqrt{x}$ kg per m^3. Find a formula involving definite integrals that computes the mass of T.

Solution: Since the density depends on the variable x, you need to take slices perpendicular to the x-axis. Hence

$$m = \int_0^2 (2 - \sqrt{x})\pi[(x+2)^2 - x^4]dx.$$

c. [2 points] Find a formula involving definite integrals that computes the value of $\bar{x}$, the x coordinate of the center of mass of the solid T.

Solution:
$$\frac{\int_0^2 x(2 - \sqrt{x})\pi[(x+2)^2 - x^4]dx}{\int_0^2 (2 - \sqrt{x})\pi[(x+2)^2 - x^4]dx}.$$