Math 116 — Second Midterm

March 20, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	13	
3	10	
4	13	
5	12	
6	11	
7	15	
8	14	
Total	100	

1. [12 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Consider the parametric equation given by $x = a(1+t^2)$ and $y = 1 - t^3$, where $a > 0$. Then the curve is concave up at the point $(x, y) = (2a, 0)$.

True $\boxed{\text{False}}$

Solution: For $t \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)} = \frac{\frac{d}{dt}\left(\frac{-3t^2}{2at}\right)}{2at} = \frac{\frac{d}{dt}\left(\frac{-3t^2}{2at}\right)}{2at} = \frac{\frac{-3}{2a}}{2at} = \frac{-3}{4a^2t}.$$

Since $(x(1), y(1)) = (2a, 0)$, then $\frac{d^2y}{dx^2}|_{t=1} = \frac{-3}{4a^2} < 0$ at that point. Hence the curve is concave down at $(2a, 0)$.

b. [2 points] Let $f(x)$ be a continuous function satisfying $\lim_{x\to\infty} f(x) = 0$. Then

$$\lim_{b\to\infty}\int_b^\infty f(x)dx = 0.$$

True $\boxed{\text{False}}$

Solution: If $\int_0^\infty f(x)dx$ diverges, then for any $b > 0$, $\int_b^\infty f(x)dx$ diverges. Then $\lim_{b\to\infty}\int_b^\infty f(x)dx \neq 0$.

c. [2 points] The point P whose polar coordinates $(r, \theta) = (1, \frac{\pi}{6})$ also has coordinates $(r, \theta) = (-1, \frac{7\pi}{6})$.

$\boxed{\text{True}}$ False

Solution:

d. [2 points] $\int_0^2 \ln(1+t)\, dt$ is an improper integral.

True $\boxed{\text{False}}$

Solution: The function $\ln(1+t)$ is continuous on $[0, 2]$.

e. [2 points] All the solutions $y(t)$ of the differential equation $\frac{dy}{dt} = t^3$ are concave up.

$\boxed{\text{True}}$ False

Solution: The solution $y(t)$ satisfies $\frac{d^2y}{dt^2} = 3t^2 \geq 0$ hence concave up.

f. [2 points] The length of the parametric curve given by $x = \cos t$ and $y = \cos t + 1$ is $2\sqrt{2}$.

$\boxed{\text{True}}$ False

Solution: The length of the curve is L

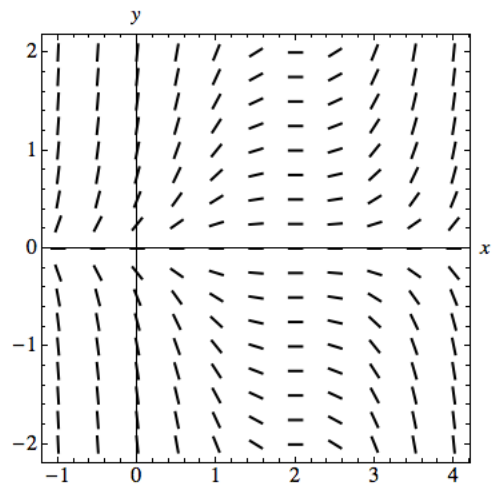
$$L = \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^\pi \sqrt{(-\sin t)^2 + (-\sin t)^2}\, dt = \sqrt{2}\int_0^\pi \sin t \; dt = 2\sqrt{2}.$$

2. [13 points]

a. [7 points] Consider the following differential equations:

A. $y' = y(x-2)^2$ B. $y' = y(x-2)$ C. $y' = -y(1-y)$ D. $y' = -y^2(1-y)$

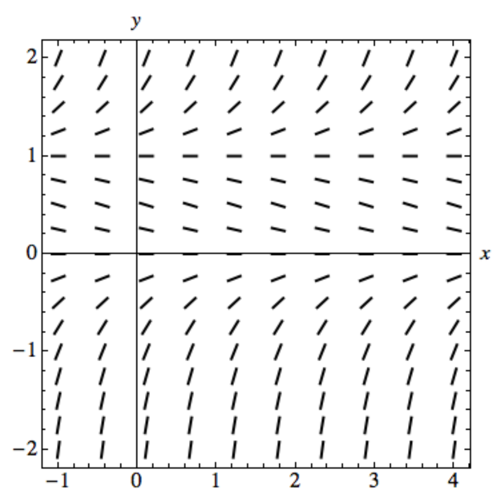
Each of the following slope fields belongs to one of the differential equations listed above. Indicate which differential equation on the given line. Find the equation of the equilibrium solutions and their stability. If a slope field has no equilibrium solutions, write none.



Differential equation: **A**

Equilibrium solutions and stability:

$y = 0$ **unstable**.



Differential equation: **C**

Equilibrium solutions and stability:

$y = 1$ **unstable**.

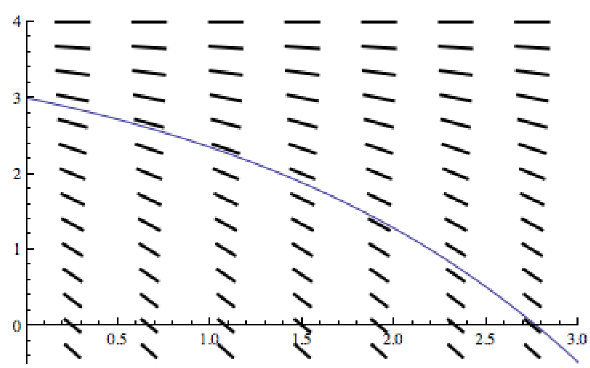
$y = 0$ **stable**.

b. [4 points] A bank account earns a p percent annual interest compounded continuously. Continuous payments are made out of the account at a rate of q thousands of dollars per year. Let $B(t)$ be the amount of money (**in thousands of dollars**) in the account t years after the account was opened. Write the differential equation satisfied by $B(t)$.

Solution:

$$\frac{dB}{dt} = \frac{p}{100}B - q.$$

c. [2 points] The slope field shown below corresponds to the differential equation satisfied by $B(t)$ (for certain values of p and q). Sketch on the slope field below the solution to the differential equation that corresponds to an account opened with an initial deposit of $3,000$ dollars.



Solution:

3. [10 points] The function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} + 2y = 2t \qquad \text{with} \qquad y(0) = 1.$$

a. [1 point] Can you use the method of separation of variables to solve this differential equation?

Solution: No.

b. [4 points] Use Euler's method with two steps to estimate $y(1)$. Fill the table with the values you find.

t	0		
$y(t)$			

Solution: The equation can be rewritten as $\frac{dy}{dt} = 2t - 2y$

$$y(0) = 1 \quad \text{and} \quad \Delta t = \frac{1}{2}.$$
$$y(\frac{1}{2}) \approx 1 + (2(0) - 2(1))\frac{1}{2} = 0.$$
$$y(1) \approx 0 + (2(0.5) - 2(0))\frac{1}{2} = \frac{1}{2}.$$

t	0	0.5	1
$y(t)$	1	0	$\frac{1}{2}$

c. [5 points] For what values of a and b is the function $y(t) = ae^{-2t} + b + t$ a solution to the differential equation

$\frac{dy}{dt} + 2y = 2t$ with $y(0) = 1$ $\qquad a =$ ____________ $\qquad b =$ ____________.

Solution: Using the initial condition $y(0) = 1$, you get $a + b = 1$.
Plugging into the differential equation

$$\frac{dy}{dt} + 2y = (-2ae^{-2t} + 1) + 2(ae^{-2t} + b + t) = 1 + 2b + 2t.$$
$$\frac{dy}{dt} + 2y = 2t \text{ implies}$$
$$1 + 2b + 2t = 2t.$$
$$1 + 2b = 0 \text{ then } b = -\frac{1}{2}$$

.

Hence $b = -\frac{1}{2}$ and $a = 1.5$.

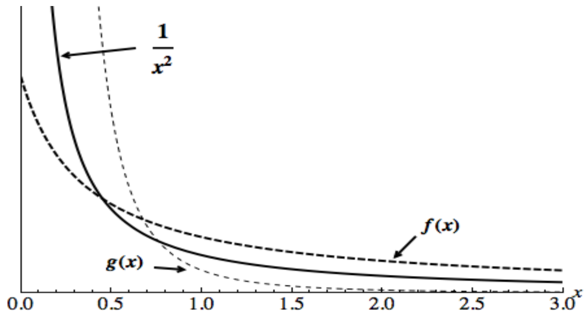
4. [13 points]

a. [8 points] Consider the functions $f(x)$ and $g(x)$ where

$$\frac{1}{x^2} \le g(x) \qquad \text{for} \qquad 0 < x < \frac{1}{2}.$$

$$g(x) \le \frac{1}{x^2} \qquad \text{for} \qquad 1 < x$$

$$\frac{1}{x^2} \le f(x) \qquad \text{for} \qquad 1 < x.$$



Using the information about $f(x)$ and $g(x)$ provided above, determine which of the following integrals is convergent or divergent. Circle your answers. If there is not enough information given to determine the convergence or divergence of the integral circle NI.

i) $\displaystyle\int_1^\infty f(x)dx$ CONVERGENT DIVERGENT **NI**

ii) $\displaystyle\int_1^\infty g(x)dx$ **CONVERGENT** DIVERGENT NI

iii) $\displaystyle\int_0^1 f(x)dx$ **CONVERGENT** DIVERGENT NI

iv) $\displaystyle\int_0^1 g(x)dx$ CONVERGENT **DIVERGENT** NI

b. [5 points] Does $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}dx$ converge or diverge? If the integral converges, compute its value. Show all your work. Use u substitution.

Solution:

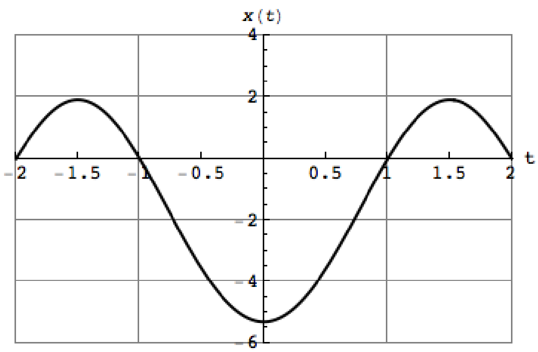
$$\int_e^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}dx$$

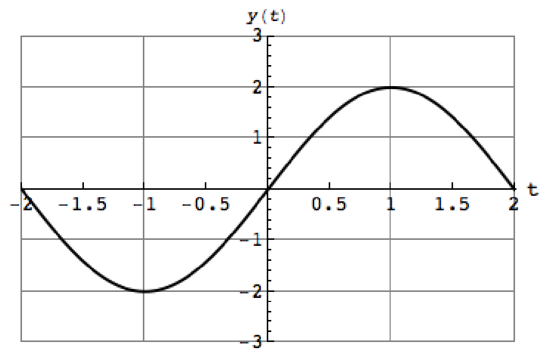
$$\text{using } u = \ln x \qquad = \lim_{b\to\infty}\int_1^{\ln b}\frac{1}{u^2}dx = \lim_{b\to\infty} -\frac{1}{u}\Big|_1^b = \lim_{b\to\infty} 1 - \frac{1}{\ln b} = 1 \qquad \text{converges.}$$

5. [12 points] A particle moves according to the following parametric equations

$$x = x(t) \quad \text{and} \quad y = y(t) \quad \text{for} \quad -2 \le t \le 2,$$

where the graphs of $x(t)$ and $y(t)$ are shown below.





a. [2 points] Is there a value of t at which the particle is at the point $(0, 2)$? If so, find the value of t where this happens.

Solution: $t = 1$.

b. [3 points] At which value(s) of t is the particle on the x-axis?

Solution: $t = -2, 0, 2$.

c. [4 points] At what points (x, y) does the curve traveled by the particle have a horizontal tangent line? Include the times for each point.

Solution: $y'(t) = 0$ when $t = 1$, $(x, y) = (0, 2)$ and $t = -1$, $(x, y) = (0, -2)$.

d. [3 points] For which of values of t is the slope of the tangent line to the curve positive?

Solution: Slope$=\frac{y'(t)}{x'(t)} > 0$ if x' and y' have the same sign. This occurs at $(0, 1)$, $(-1.5, -1)$ and $(1.5, 2)$.

6. [11 points]

a. [8 points] Use the **comparison method** to determine the convergence or divergence of the following improper integrals. Justify your answers. Make sure to properly cite any results of convergence or divergence of integrals that you use.

i) $\displaystyle\int_1^\infty \frac{3+\sin(4x)}{\sqrt[3]{x}}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^{1/3}}$. Because $3+\sin(4x) \geq 2$, we know that

$$\frac{3+\sin(4x)}{x^{1/3}} \geq \frac{2}{x^{1/3}}.$$

By the p-test with $p = 1/3$, we know that $\int_1^\infty \frac{1}{x^{1/3}}dx$ diverges. Therefore, by the comparison method, we know that this integral diverges, too.

ii) $\displaystyle\int_4^\infty \frac{1}{\sqrt{x}+x^2}dx.$

Solution: We compare the integrand with the function $\frac{1}{x^2}$. Because $\sqrt{x} \geq 0$, we know that

$$\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{x^2}.$$

By the p-test with $p = 2$, we know that $\int_4^\infty \frac{1}{x^2}dx$ converges. Therefore, by the comparison test, $\int_4^\infty \frac{1}{\sqrt{x}+x^2}$ converges, too.

b. [3 points] For which values of p does the following integral converges?

$$\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$$

No justification is required.

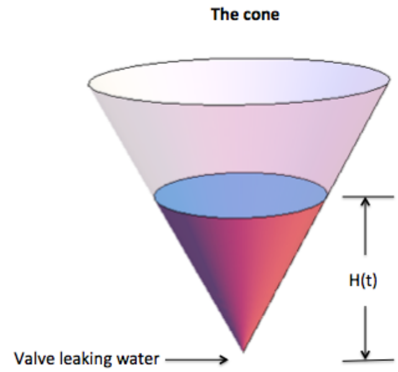
Solution: If $p \leq 2$, the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{1}{4}$ for large values of x. Hence the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx.$ diverges.

If $p > 2$, then the function $\displaystyle\frac{x^2-1}{x^p+4x^2+2}dx$ behaves as the function $\displaystyle\frac{x^2}{x^p} = \frac{1}{x^{p-2}}$ for large values of x. Then $\displaystyle\int_2^\infty \frac{1}{x^{p-2}}dx$ converges if $p > 3$ $(p-2 > 1)$. Therefore the integral $\displaystyle\int_2^\infty \frac{x^2-1}{x^p+4x^2+2}dx$ converges for $p > 3$.

7. [15 points] A cone is filled with water up to a depth of H_0 m. At time $t = 0$, a valve at the bottom of the cone is opened. Water leaks out of the cone through the opened valve. Let $H(t)$ be the depth of the water (in m) in the cone at time t (in hours).

The function $H(t)$ satisfies the differential equation

$$\frac{dH}{dt} = \frac{k}{H^{\frac{3}{2}}}$$



a. [2 points]
What must be the sign and units of k?

Solution: The sign of k is negative, because the water is dripping out. Because dH/dt is in meters per hour and $h^{3/2}$ is in $\text{m}^{3/2}$, we know that k must have units $\text{m}^{5/2}$ per hour.

b. [7 points] Find a formula for $H(t)$. Your formula should include k and H_0

Solution: We use separation of variables.

$$\int H^{3/2} dH = \int k dt$$

$$\frac{H^{5/2}}{5/2} = kt + C_0$$

$$H^{5/2} = \frac{5}{2}kt + C_1$$

$$H = \left(\frac{5}{2}kt + C_1\right)^{2/5}$$

Since

$$H_0 = H(0) = C_1^{2/5}$$

we know that $C_1 = H_0^{5/2}$. Therefore,

$$H(t) = \left(\frac{5}{2}kt + H_0^{5/2}\right)^{2/5}.$$

This problem continues on the next page.

Problem 7 continued

c. [4 points] If the cone is filled with water up to a depth of 4 m at $t = 0$. What should the value of k be in order for the cone to be empty after an hour? Show all your work.

Solution: We know $H_0 = 4$. Therefore, the equation is $(5/2kt + 32)^{2/5} = H(t)$. Setting this equal to zero, we have

$$\left(\frac{5}{2}k + 32\right)^{2/5} = 0$$

So we want to solve

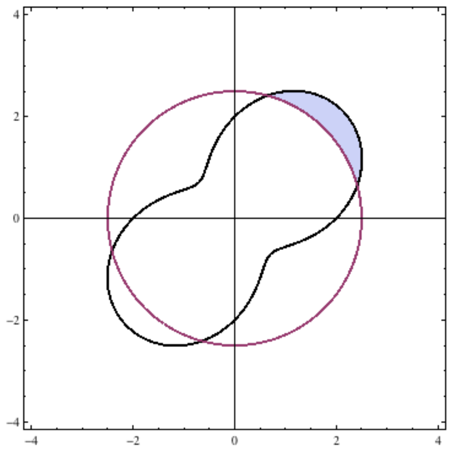
$$\frac{5}{2}k + 32 = 0 \Rightarrow k = -\frac{64}{5}.$$

d. [2 points] Does the differential equation satisfied by H have equilibrium solutions? If it does, find them.

Solution: No

8. [14 points]

a. [6 points] Find a definite integral that computes the shaded area outside the circle $r = \frac{5}{2}$ and inside the curve given by $r = 2 + \sin 2\theta$ in the graph below.



Solution:

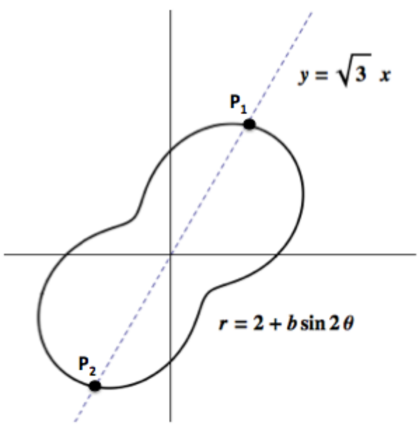
$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((2+\sin(2\theta))^2 - \left(\frac{5}{2}\right)^2 \right) d\theta.$$

Here we found the limits of integration by solving for where $5/2 = 2 + \sin 2\theta$ for θ in the first quadrant.

b. [4 points] Find the polar coordinates (r, θ) of the points where the line $y = \sqrt{3}\,x$ intersects the graph of $r = 2 + b\sin 2\theta$. Here the constant $0 < b < 2$. Your answers may include b.

P_1=____________________

P_2=____________________



Solution: We want $\tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. In the first case, we have

$$P_1 = (r,\theta) = \left(2 + b\sin\left(\frac{2\pi}{3}\right), \frac{\pi}{3}\right) = \left(2 + \frac{\sqrt{3}}{2}b, \frac{\pi}{3}\right).$$

$$P_2 = (r,\theta) = \left(2 + b\sin\left(\frac{8\pi}{3}\right), \frac{4\pi}{3}\right) = \left(2 + \frac{\sqrt{3}}{2}b, \frac{4\pi}{3}\right).$$

c. [4 points]

i) Find the equation in polar coordinates of the line $x = 0$.

Solution: $\theta = \dfrac{\pi}{2}$.

ii) Find the equation in polar coordinates of the line $y = 4$.

Solution: $y = r\sin\theta = 4$, so $r = \dfrac{4}{\sin\theta}$.