Math 116 — Final Exam

April 26, 2013

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	6	
3	10	
4	10	
5	14	
6	10	
7	9	
8	8	
9	14	
10	9	
Total	100	

You may find the following expressions useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

1. [10 points] Indicate if each of the following is true or false by circling the correct answer. No justification is required.

a. [2 points] Let $-1 < q < 1$, then

$$\sum_{n=1}^{\infty} q^n = q + q^2 + q^3 + \cdots + q^n + \cdots = \frac{q}{1-q}.$$

True False

Solution: Since $\sum_{n=1}^{\infty} q^n = q\left(\sum_{n=0}^{\infty} q^n\right)$, then using the formula for geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ with $r = q$ yields the result.

b. [2 points] Let $F(t)$ be an antiderivative of a continuous function $f(t)$. If the units of $f(t)$ are meters and t is in seconds, then the units of $F(t)$ are meters per second.

True **False**

Solution: The Second Fundamental Theorem of Calculus says that if $F(t)$ is an antiderivative of $f(t)$, then $F(t) = \int_a^t f(x)dx$. The units of a definite integral are the unites of $f(t)$ times the units of t. In this case, the units of $F(t)$ are meters times seconds.

c. [2 points] If the motion of a particle is given by the parametric equations

$$x = \frac{at}{1+t^3}, \quad y = \frac{at^2}{1+t^3} \qquad \text{for} \quad a > 0,$$

then the particle approaches the origin as t goes to infinity.

True False

Solution: Since $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, then the particle approaches the origin as t goes to infinity.

d. [2 points] Let a_n be a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = \infty$. Then the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

True **False**

Solution: If $a_n = n$, then $\lim_{n\to\infty} n = \infty$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test.

e. [2 points] Let $f(x)$ be a continuous function. Then

$$\int_0^1 f(2x)dx = \frac{1}{2}\int_0^1 f(x)dx.$$

True | False |

Solution: Using the substitution $u = 2x$, you get

$$\int_0^1 f(2x)dx = \frac{1}{2}\int_0^2 f(u)du.$$

2. [6 points] Let the sequence a_n be given by

$$a_1 = -1, \quad a_2 = \frac{\sqrt{2}}{3}, \quad a_3 = -\frac{\sqrt{3}}{5}, \quad a_4 = \frac{\sqrt{4}}{7}, \quad a_5 = -\frac{\sqrt{5}}{9}, \quad a_6 = \frac{\sqrt{6}}{11}$$

a. [1 point] Find a_7.

Solution:

$$a_7 = \frac{-\sqrt{7}}{13}.$$

b. [3 points] Write a formula for a_n.

Solution:

$$a_n = (-1)^n \frac{\sqrt{n}}{2n-1}.$$

c. [2 points] Does the sequence a_n converge? If so, find its limit.

Solution: Yes, it converges to 0.

3. [10 points] A boat's initial value is $100,000; it loses 15% of its value each year. The boat's maintenance cost is $500 the first year and increases by 10% annually.
In the following questions, your formulas should not be recursive.

a. [2 points] Let B_n be the value of the boat n years after it was purchased. Find B_1 and B_2.

Solution:
$B_1 = \$100,000(0.85)$.
$B_2 = \$100,000(0.85)^2$.

b. [3 points] Find a formula for B_n.

Solution: $B_n = 100,000(.85)^n$

c. [2 points] Let M_n be the total amount of money spent on the maintenance of the boat during the first n years. Find M_2 and M_3.

Solution:
$M_2 = 500(1 + 1.1)$
$M_3 = 500(1 + 1.1 + (1.1)^2)$

d. [3 points] Find a closed form formula for M_n.

Solution: $M_n = 500\dfrac{(1-(1.1)^n)}{1-1.1}$

4. [10 points] The lifetime t (in years) of a tree has probability density function

$$f(t) = \begin{cases} \dfrac{a}{(t+1)^p} & \text{for} \quad t \geq 0. \\ 0 & \text{for} \quad t < 0. \end{cases}$$

where $a > 0$ and $p > 1$.

a. [4 points] Use the comparison method to find the values of p for which the average lifetime M is finite ($M < \infty$). Properly justify your answer.

Solution: The average lifetime M is given by the formula $M = \int_0^\infty t\frac{a}{(t+1)^p}dt$.

If $p > 2$: Since

$$t\frac{a}{(t+1)^p} \leq t\frac{a}{t^p} = \frac{a}{t^{p-1}} \qquad \text{for } t > 1$$

and $\int_1^\infty \frac{a}{t^{p-1}}dt$ converges by the p-test (since $p-1 > 1$), $\int_1^\infty t\frac{a}{(t+1)^p}dt$ converges by the Comparison Test.

If $p \leq 2$: Since $\frac{1}{t+1} > \frac{1}{2t}$ when $t > 1$, we have

$$t\frac{a}{(t+1)^p} \geq t\frac{a}{(2t)^p} = \frac{a}{2^p t^{p-1}} \qquad \text{for } t > 1.$$

Since $\int_1^\infty \frac{1}{2^p t^{p-1}}dt$ diverges by the p-test (since $p-1 \leq 1$), it follows that $\int_1^\infty \frac{a}{t^{p-1}}dt$ diverges by the Comparison Test.

b. [4 points] Find a formula for a in terms of p. Show all your work.

Solution: We know that

$$1 = \int_0^\infty \frac{a}{(t+1)^p}dt.$$

We use u-substition with $u = t+1$ to calculate the integral:

$$\begin{aligned} \int_0^\infty \frac{a}{(t+1)^p}dt &= \lim_{b\to\infty}\int_0^b \frac{a}{(t+1)^p}dt = \lim_{b\to\infty}\int_1^{b+1}\frac{a}{u^p}du = a\lim_{b\to\infty}\int_1^{b+1} u^{-p}du \\ &= a\lim_{b\to\infty}\frac{u^{-p+1}}{(-p+1)}\Big|_1^{b+1} = a\lim_{b\to\infty}\frac{1}{(-p+1)u^{p-1}}\Big|_1^{b+1} \\ \text{(since } p > 1) \quad &= \frac{a}{p-1}. \end{aligned}$$

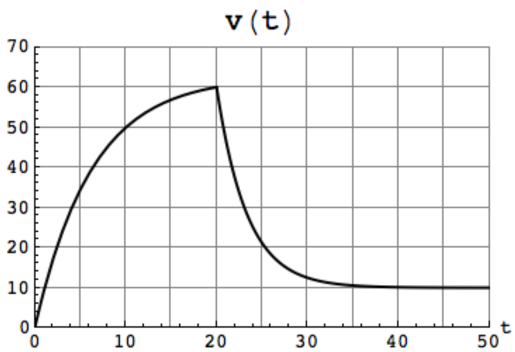
Therefore $1 = \frac{a}{p-1}$, so $a = p - 1$.

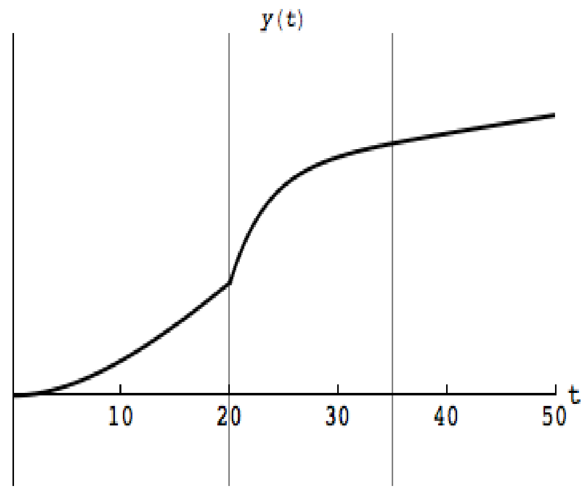
c. [2 points] Let $C(t)$ be the cumulative distribution function of $f(t)$. For a given tree, what is the practical interpretation of the expression $1 - C(30)$?

Solution: $1 - C(30)$ is the probability that a given tree lives at least 30 years.

5. [14 points] A skydiver jumps from a plane at a height of 2,000 meters above the ground. After some time in free-fall, he opens his parachute, reducing his speed, and lands safely on the ground.

a. [5 points] The graph of the skydiver's downward velocity $v(t)$ (in meters per second) t seconds after he jumped is shown below.
Sketch the graph of the antiderivative $y(t)$ of $v(t)$ satisfying $y(0) = 0$. Make sure your graph reflects the regions at which the function is increasing, decreasing, concave up or concave down.





It is important to notice that $y'(t)$ exist for all values of t since $y'(t) = v(t)$.

b. [3 points] Write down a right-hand sum with 4 subintervals in order to approximate the **average** downward velocity of the skydiver during the time the skydiver is in free-fall. Show all the terms in your sum.

Solution: The average downward velocity is $\frac{1}{20}\int_0^{20} v(t)dt$. We approximate this as

$$\frac{1}{20}\int_0^{20} v(t)dt \approx \frac{5(35+50+55+60)}{20}$$

c. [2 points] Is your estimate in (**b**) guaranteed to be an underestimate or overestimate of the average velocity of the skydiver, or there is not enough information to decide? Justify.

Solution: It's guaranteed to be an overestimate, because $v(t)$ is increasing throughout $[0, 20]$.

d. [4 points] Find a formula for the height $H(t)$ (in meters) above the ground of the skydiver t seconds after he jumped.

Solution: $H(t) = 2,000 - \int_0^t v(s)ds = 2,000 - y(t)$.

6. [10 points] At a hospital, a patient is given a drug intravenously at a constant rate of r mg/day as part of a new treatment. The patient's body depletes the drug at a rate proportional to the amount of drug present in his body at that time. Let $M(t)$ be the amount of drug (in mg) in the patient's body t days after the treatment started. The function $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = r - \frac{1}{4}M \quad \text{with} \quad M(0) = 0.$$

a. [7 points] Find a formula for $M(t)$. Your answer should depend on r.

Solution: We use separation of variables

$$\frac{dM}{r - \frac{1}{4}M} = dt.$$

Using u-substition with $u = r - 1/4M, du = -1/4dM$ for the left-hand-side, we anti-differentiate:

$$-4 \ln|r - \frac{1}{4}M| = t + C_1.$$

Therefore,

$$\ln|r - \frac{1}{4}M| = -t/4 + C_2$$

and

$$|r - \frac{1}{4}M| = e^{-t/4+C_2} = C_3 e^{-t/4}.$$

Therefore

$$1/4M = r - C_3 e^{-t/4}$$

and

$$M(t) = 4r - C_4 e^{-t/4}.$$

With $M(0) = 0$, we conclude that $C_4 = 4r$, so we get $M(t) = 4r - 4re^{-t/4}$.

b. [1 point] Find all the equilibrium solutions of the differential equation.

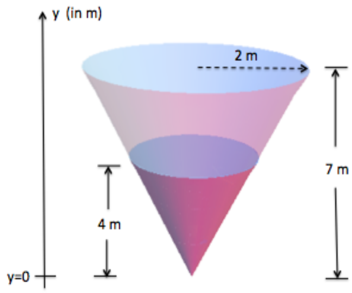
Solution: $M = 4r$.

c. [2 points] The treatment's goal is to stabilize in the long run the amount of drug in the patient at a level of 200 mg. At what rate r should the drug be administered?

Solution: You need $4r = 200$, then $r = 50$ mg/day.

7. [9 points] A tank has the shape of a circular cone. The cone has radius 2 m and height 7 m (as shown below). The tank contains a liquid up to a depth of 4 m. The density of the liquid is $\delta(y) = 1100 - y^2$ kg/m^3, where y measures the distance in meters from the bottom of the tank. Use the value $g = 9.8$ m/s^2 for the acceleration due to gravity.

a. [6 points] Find a definite integral that computes the mass of the liquid in the tank. Show all your work.



Solution: Let $r(y)$ be the radius at height y. By similar triangles, $2/7 = r/y$, so $r = \frac{2}{7}y$. The approximate mass of a thin slice at height y is $\pi(2/7y)^2(1100-y)^2\Delta y$, so the answer is

$$\int_0^4 \pi(2/7y)^2(1100 - y^2)dy.$$

b. [3 points] Find a definite integral that computes the work required to pump the liquid 2 meters above the top of the tank. Show all your work.

Solution: We want to lift each thin slice $(9 - y)$ feet. The work to lift a slice is $9.8(9 - y)\pi(2/7y)^2(1100 - y^2)\Delta y$, so the integral is

$$\int_0^4 9.8(9 - y)\pi(2/7y)^2(1100 - y^2)dy.$$

8. [8 points] Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-5)^n.$$

In the following questions, you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Show all your work.

a. [2 points] Does the series converge or diverge at $x = 3$?

Solution: At $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the alternating series test, since $1/\sqrt{n}$ is decreasing and converges to 0.

b. [2 points] What does your answer from part (**a**) imply about the radius of convergence of the series?

Solution: Because it converges at $x = 3$, we know that the radius of convergence $R \geq 2$.

c. [4 points] Find the interval of convergence of the power series.

Solution: Using the ratio test, we have

$$\lim_{n\to\infty} \frac{\frac{1}{2^{n+1}\sqrt{n+1}}|x-5|^{n+1}}{\frac{1}{2^n\sqrt{n}}|x-5|^n} = \frac{1}{2}|x-5| = L,$$

so the radius of convergence is 2. Now we have to check the endpoints. We know from part (a) that it converges at $x = 3$. For $x = 7$, we get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Thus, the interval of convergence is $3 \leq x < 7$.

9. [14 points] Determine the convergence or divergence of the following series. In questions (**a**) and (**b**) you need to support your answers by stating and properly justifying the use of the test(s) or facts you used to prove the convergence or divergence of the series. Circle your answer. Show all your work.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+1}}$ Converges Diverges

Solution: You can use either the limit comparison test or the comparison test. We simply use the comparison test. We know that

$$0 < \frac{2n}{\sqrt{n^5+1}} \le \frac{2n}{n^{5/2}} \le 2\frac{1}{n^{3/2}}.$$

Because $\sum \frac{1}{n^{3/2}}$ converges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^5+1}}$ converges by the comparison test.

b. [4 points] $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ Converges Diverges

Solution: Since the function $f(x) = x^2 e^{-x^3}$ is positive and decreasing for $x > 1$, we can use the integral test to determine the convergence or divergence of $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$. To do this, we use u-substitution. Let $u = -x^3$, $du = -3x^2 dx$. Therefore

$$\int_0^{\infty} x^2 e^{-x^3} dx = \lim_{b\to\infty} \int_0^b x^2 e^{-x^3} dx = \lim_{b\to\infty} \frac{1}{3} \int_{-b^3}^{0} e^u du$$
$$= \lim_{b\to\infty} \frac{1}{3} e^u \Big|_{-b^3}^{0} = \frac{1}{3}.$$

Hence $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by the integral test.

c. [6 points] Determine if the following series converge absolutely, conditionally or diverge. Circle your answers. No justification is required.

a). $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(3n)}{n^6+1}$

Converges absolutely Converges conditionally Diverges

b). $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+1}$

Converges absolutely Converges conditionally **Diverges**

10. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series of $f(y) = \dfrac{1}{(1+y)^{\frac{3}{2}}}$ about $y = 0$. Show all your work.

Solution: Using the binomial expansion, this is

$$\frac{1}{(1+y)^{3/2}} \approx 1 - \frac{3}{2}y + \frac{(-3/2)\cdot(-5/2)}{2}y^2 = 1 - \frac{3}{2}y + \frac{15}{8}y^2$$

b. [2 points] Use your answer in (**a**) to find the first three nonzero terms in the Taylor series of $g(x) = \dfrac{1}{(a^2+x^2)^{\frac{3}{2}}}$ about $x = 0$. Show all your work.

Solution: Factoring, we have

$$\frac{1}{(a^2+x^2)^{\frac{3}{2}}} = \frac{1}{(a^2(1+(\frac{x}{a})^2))^{\frac{3}{2}}} = \frac{1}{(a^2)^{\frac{3}{2}}(1+(\frac{x}{a})^2)^{\frac{3}{2}}} = \frac{1}{a^3(1+(\frac{x}{a})^2)^{3/2}}.$$

Therefore, letting $y = \left(\frac{x}{a}\right)^2$, we have

$$\frac{1}{a^3(1+(\frac{x}{a})^2)^{3/2}} = \frac{1}{a^3}\left(\frac{1}{(1+y)^{3/2}}\right) \approx \frac{1}{a^3}\left(1 - \frac{3}{2}y + \frac{15}{8}y^2\right)$$

$$= \frac{1}{a^3}\left(1 - \frac{3}{2}\left(\frac{x}{a}\right)^2 + \frac{15}{8}\left(\frac{x}{a}\right)^4\right) = \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.$$

$$\frac{1}{(a^2+x^2)^{\frac{3}{2}}} \approx \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4.$$

c. [2 points] For which values of x is the Taylor series for $g(x)$ about $x = 0$ expected to converge?

Solution: The Binomial series in a) converges for $|y| < 1$. This implies that the series for $g(x)$ converges for all values of x satisfying $|\frac{x}{a}| < 1$, so $-|a| < x < |a|$.

Problem continues on the next page

Continuation of problem 10.

The force of gravitational attraction F between a rod of length $2L$ and a particle at a distance a is given by

$$F = k\int_0^L \frac{1}{(a^2+x^2)^{\frac{3}{2}}}dx,$$

where k is a positive constant.

d. [2 points] Use your answer in (**b**) to obtain an approximation for the force of gravitational attraction F between the rod and the particle. Your answer should depend on the constants k, a and L. Show all your work.

Solution: We want

$$F = k\int_0^L \frac{1}{(a^2+x^2)^{3/2}}dx \approx k\int_0^L \frac{1}{a^3} - \frac{3}{2a^5}x^2 + \frac{15}{8a^7}x^4 dx$$
$$= k\left(\frac{L}{a^3} - \frac{1}{2a^5}L^3 + \frac{3}{8a^7}L^5\right).$$

Hence $F \approx \dfrac{kL}{a^3} - \dfrac{k}{2a^5}L^3 + \dfrac{3k}{8a^7}L^5$.