Math 116 — First Midterm

February 10, 2014

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	7	
2	5	
3	15	
4	9	
5	10	
6	11	
7	8	
8	12	
9	12	
10	6	
11	5	
Total	100	

1. [7 points] The table below gives values of a function, $f(x)$, at several points.

x	4	5	6	7	8
$f(x)$	3	5	4	1	2

a. [3 points] Estimate the integral $\int_4^8 f(x)dx$ using Mid(2). Be sure to write out all the terms of your sum.

Solution:
Mid(2) $= 2(f(5) + f(7)) = 2(5 + 1) = 12$.

b. [4 points] Simplify the integral $\displaystyle\int_{\ln(4)}^{\ln(7)} e^x f(e^x)dx$ and estimate the resulting integral using Trap(3). Be sure to show how you simplified the integral and to write out all the terms of your sum.

Solution:
Let $u = e^x$ then $du = e^x dx$. Changing the bounds of integration upper bound$= e^{\ln(7)} = 7$, lower bound$= e^{\ln(4)} = 4$. Thus $\displaystyle\int_{\ln(4)}^{\ln(7)} e^x f(e^x)dx = \int_4^7 f(u)du.$
Trap(3) $= \frac{1}{2}$(Left(3)+Right(3)) $= \frac{1}{2}f(4) + f(5) + f(6) + \frac{1}{2}f(7) = 11$.

2. [5 points] Suppose that $g(x) = w(x)v(x)$ where the functions $w(x)$ and $v(x)$ are both positive, decreasing and concave down on the interval $[0, 1]$.

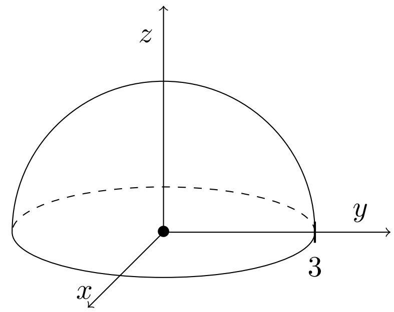
a. [2 points] Write the derivatives $g'(x)$ and $g''(x)$ in terms of $w(x)$, $v(x)$, and their derivatives.

Solution:
$g'(x) = w'(x)v(x) + w(x)v'(x)$
$g''(x) = w''(x)v(x) + 2w'(x)v'(x) + w(x)v''(x)$

b. [3 points] Circle the method(s) that will ALWAYS UNDERESTIMATE the integral $\int_0^1 g(x)dx$.

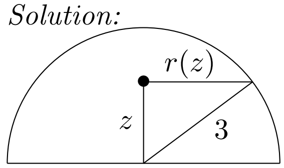
Left $\boxed{\text{Right}}$ Mid Trap

3. [15 points] Consider a hemisphere of radius 3m shown below. The hemisphere is filled to the top with water. The density of the water is 1000 kg/m^3.



a. [4 points] Find an expression for the mass of a circular slice of thickness Δz that is z meters above the base of the hemisphere.

Solution:



Using the Pythagorean formula $r(z) = \sqrt{9 - z^2}$. We have Mass $= 1000\pi r(z)^2 \Delta z$. Plugging in $r(z)$ we have Mass $= 1000\pi(9 - z^2)\Delta z$.

b. [7 points] What is the center of mass of the hemisphere of water? Justify your answers. Please limit any verbal explanation to a sentence or two.

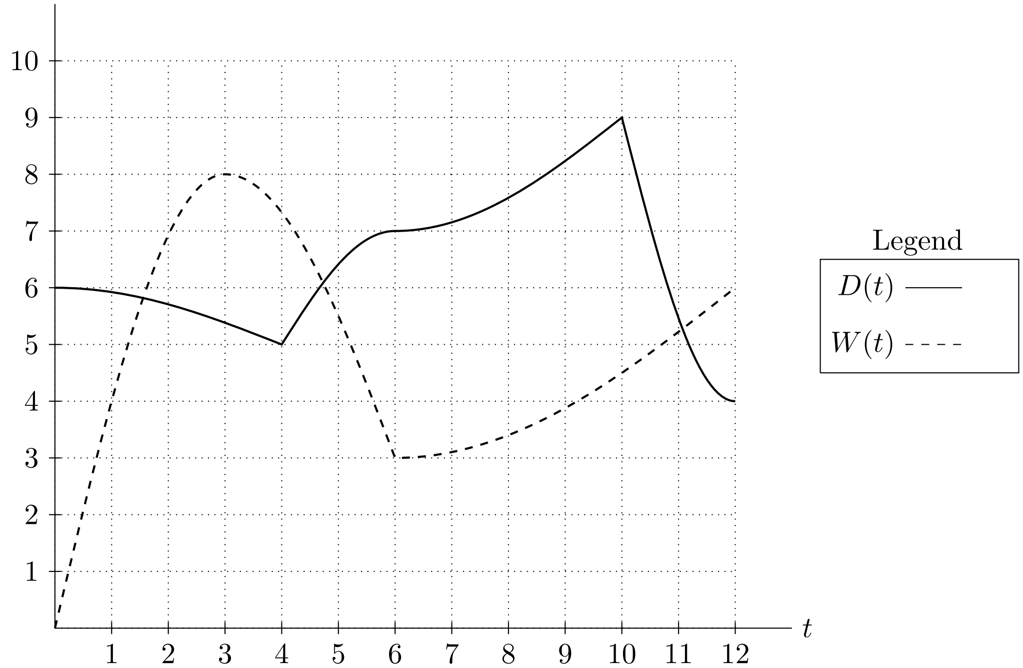
Solution:
$\bar{x} = 0$ and $\bar{y} = 0$ because density is constant and the hemisphere is symmetric about the x and y axes.

$$\bar{z} = \frac{\int_0^3 1000\pi z(9 - z^2)dz}{\int_0^3 1000\pi(9 - z^2)dz} = \frac{\int_0^3 z(9 - z^2)dz}{\int_0^3 (9 - z^2)dz} = 9/8.$$

c. [4 points] Suppose water is evaporating from the hemisphere and the height of the water is decreasing at a constant rate of 1 m/day. Assuming $0 \leq t < 3$, write an expression involving integrals which gives the z-coordinate of the center of mass of the water, t days after the water started evaporating. Do not evaluate any integrals.

Solution: The height of the water at time t is $3 - t$. Thus we now integrate from 0 to $3 - t$. $\bar{z} = \dfrac{\int_0^{3-t} 1000\pi z(9 - z^2)dz}{\int_0^{3-t} 1000\pi(9 - z^2)dz} = \dfrac{\int_0^{3-t} z(9 - z^2)dz}{\int_0^{3-t}(9 - z^2)dz}$

4. [9 points] A Swiss bank is constantly receiving deposits and withdrawals of money. Let $D(t)$ be the deposit rate (the rate at which money is going into the bank) and $W(t)$ be the withdrawal rate (the rate at which money is being taken out of the bank), both in millions of dollars/month, where t is measured in months since January 1st 2013. Suppose that on January 1st 2013 the bank has \$50 million. A graph of the two functions is shown below.



a. [4 points] Write an expression that gives the amount of money in the bank at time t. Include units.

Solution: $M(t) = \int_0^t (D(x) - W(x))dx + 50$ million dollars.

Alternatively $M(t) = 10^6 \int_0^t (D(x) - W(x)dx) + 5 * 10^7$ dollars

b. [3 points] Write an expression that gives the average rate of change of the amount of money in the bank, in millions of dollars per month, during the year 2013.

Solution: $\frac{1}{12} \int_0^{12} (D(t) - W(t))dt.$

c. [2 points] Estimate the date in 2013 when the bank has the most money in it. You do not need to show your work.

Solution: $t \approx 11$ or approximately December 1st 2013.

5. [10 points] Suppose that $f(x)$ and $g(x)$ are twice differentiable functions defined for all x with the following properties:

- $f(0) = g(0)$ and $f(1) = g(1)$.
- $f(x)$ and $g(x)$ are increasing.
- $f(x)$ is concave down and $g(x)$ is concave up.

For each of the following questions, circle the correct answer. No justification is necessary.

Solution: +2 if correct. CIRCLE CORRECT ANSWER IF WRONG.

a. [2 points] Which is larger, $\int_0^1 f(x)dx$ or $\int_0^1 g(x)dx$?

$\boxed{\int_0^1 f(x)dx}$ $\int_0^1 g(x)dx$ Equal Impossible to determine

b. [2 points] Which is larger, $\int_0^1 |f(x)|dx$ or $\int_0^1 |g(x)|dx$?

$\int_0^1 |f(x)|dx$ $\int_0^1 |g(x)|dx$ Equal $\boxed{\text{Impossible to determine}}$

c. [2 points] Which is larger, $\int_0^1 f'(x)dx$ or $\int_0^1 g'(x)dx$?

$\int_0^1 f'(x)dx$ $\int_0^1 g'(x)dx$ $\boxed{\text{Equal}}$ Impossible to determine

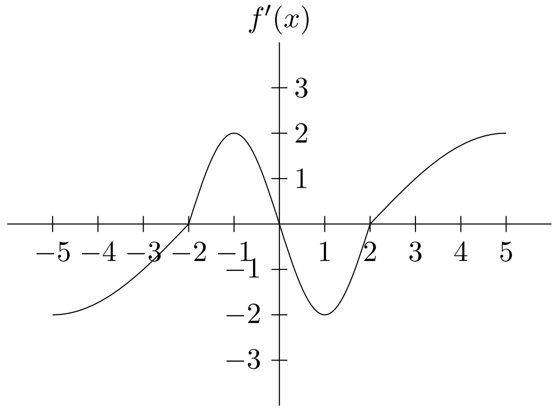
d. [2 points] Which is larger, $\int_0^1 xf'(x)dx$ or $\int_0^1 xg'(x)dx$?

$\int_0^1 xf'(x)dx$ $\boxed{\int_0^1 xg'(x)dx}$ Equal Impossible to determine

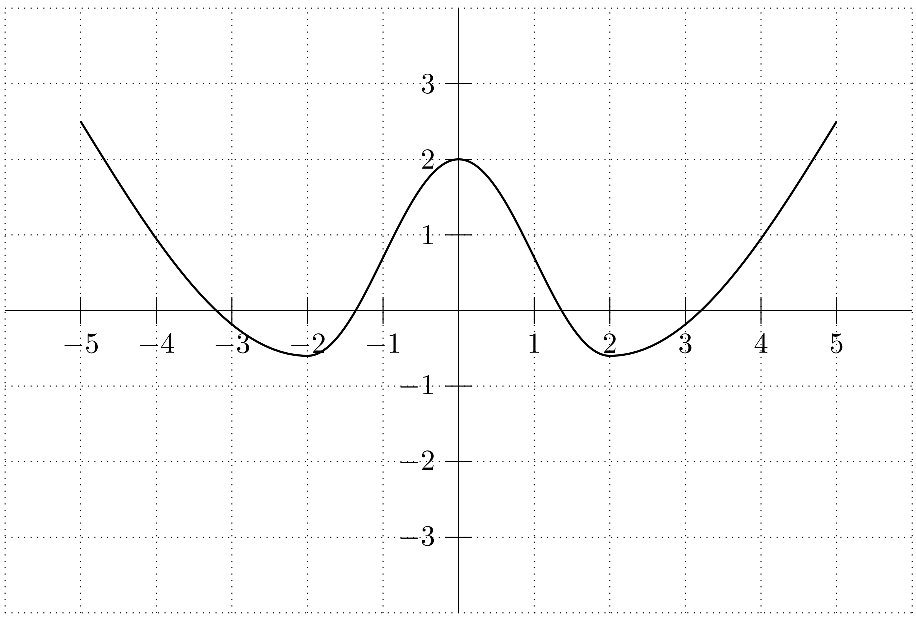
e. [2 points] Which is larger, $\int_0^1 f(x)f'(x)dx$ or $\int_0^1 g(x)g'(x)dx$?

$\int_0^1 f(x)f'(x)dx$ $\int_0^1 g(x)g'(x)dx$ $\boxed{\text{Equal}}$ Impossible to determine

6. [11 points] The graph of $f'(x)$, an odd function defined on the interval $[-5, 5]$, is shown below.



On the blank graph below, draw a graph of $f(x)$, the antiderivative of $f'(x)$ with $f(0) = 2$. Make sure your graph depicts where $f(x)$ is increasing and decreasing and the concavity of the function.



Write the x coordinates of the local minima of $f(x)$ $\underline{\quad\pm 2\quad}$

Write the x coordinates of the local maxima of $f(x)$ $\underline{\quad 0\quad}$

Write the x coordinates of the inflection points of $f(x)$ $\underline{\quad\pm 1\quad}$

7. [8 points] Alyssa Edwards wants to play a prank on Coco Montrese by spilling a bucket of orange cheese powder on her. To do this Alyssa lifts the bucket at a constant speed from the ground to a height of 10 meters. Unfortunately the bucket has a small hole and the cheese begins leaking out at a constant rate as soon as the bucket leaves the ground. The bucket initially weighs 10kg and when it reaches a height of 10 meters it only weighs 5kg. Recall the gravitational constant is $g = 9.8\text{m/s}^2$.

a. [3 points] Write an expression giving the mass of the bucket $m(h)$ when the bucket is h meters above the ground.

Solution: The bucket is being lifted and is leaking at a constant rate. Therefore the mass of the bucket at height h will be a linear function. $m(h) = 5 + (\frac{10-h}{2}) = 10 - \frac{h}{2}$

b. [5 points] How much work is required to lift the bucket from the ground to a height of 10 meters? Include units.

Solution: The force on the bucket at height h is $gm(h)$. Therefore the work is $\displaystyle\int_0^{10} gm(h)dh = \int_0^{10} g(10 - \frac{h}{2})dh = 75g = 735$ joules.

8. [12 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

a. [2 points] If $f(x)$ is positive and continuous, then $F(x) = \int_{-e^x}^{0} f(t)dt$ is increasing for all x.

$\boxed{\text{True}}$ False

b. [2 points] If $E(x)$ is an antiderivative of e^x then $\ln(E(x)) = E(\ln(x))$.

True $\boxed{\text{False}}$

c. [2 points] If $g(x)$ is concave up and increasing on $[a, b]$ then $\int_a^b g(x)dx <$ Trap(5) $<$ Right(5).

$\boxed{\text{True}}$ False

d. [2 points] If $\int_0^1 p(x)dx > \int_0^1 q(x)dx$, then $p(x) > q(x)$ for every x in $[0, 1]$.

True $\boxed{\text{False}}$

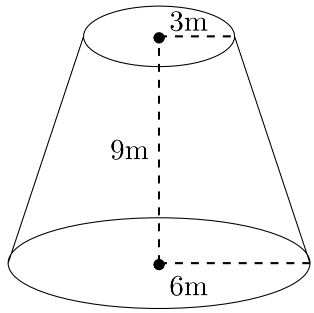
e. [2 points] If $v(x)$ is a continuous even function, then $\int_{-2}^{2} v(x)dx = \int_0^4 v(x)dx$.

True $\boxed{\text{False}}$

f. [2 points] If $f(x)$ is a continuous function, and $F(x)$ is an antiderivative of $f(x)$, then $F(x) = \int_3^x f(t)dt + K$ for some constant K.

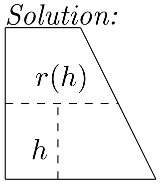
$\boxed{\text{True}}$ False

9. [12 points] The Nub's Nob Ski Area keeps a massive supply of hot chocolate. The hot chocolate is stored in a container shaped like a cone with the point end removed as shown below. The height of the container is 9 meters, and it has lower radius 6 meters and upper radius 3 meters. The hot chocolate has a density of 3000 kg/m^3. Recall the gravitational constant is $g = 9.8\text{m/s}^2$.



a. [3 points] Write a formula for $r(h)$, the radius of a circular cross section of the container h meters above the base.

Solution:



Looking at a vertical cross section of the cone we see that $r(h)$ is the width of a trapezoid at height h. The width of the trapezoid is decreasingly linearly thus $r(h)$ must be a linear function with $r(0) = 6$ and $r(9) = 3$. Therefore $r(h) = 3 + \frac{3(9-h)}{9} = 6 - h/3$.

b. [6 points] Write a formula in terms of $r(h)$ for the work required to lift a slice of hot chocolate of thickness Δh from height h to the top of the container.

Solution: The mass of the slice is $3000\pi r(h)^2 \Delta h$. The slice must be lifted $9 - h$ meters. Therefore the work to lift the slice is $3000 g\pi r(h)^2 (9 - h)\Delta h$.

c. [3 points] Write an integral that gives the work required to lift all of the hot chocolate to the top of the container. Do not evaluate this integral.

Solution: Integrating the above function from 0 to 9 the work is $\int_0^9 3000 g\pi r(h)^2 (9-h)\, dh$

10. [6 points] Suppose that $p(x)$ and $q(x)$ are functions defined on $[2,4]$ with $0 \leq p(x) < q(x) \leq 3$ for all x in $[2,4]$. Let R be the region enclosed by the graphs of $p(x), q(x)$ and the lines $x = 2$ and $x = 4$.

For the following questions circle the correct answer. You do not need to show work.

a. [3 points] What is the volume of the solid obtained by rotating R about the line $y = 5$?

A: $\int_2^4 \pi[25 - (p(x) - q(x))^2]dx$

$\boxed{\text{B: } \int_2^4 \pi[(5 - p(x))^2 - (5 - q(x))^2]dx}$

C: $\int_2^4 \pi[(5 - (q(x) - p(x)))^2]dx$

D: $\int_2^4 \pi[(5 - q(x))^2 - (5 - p(x))^2]dx$

b. [3 points] What is the volume of the solid obtained by rotating R about the line $x = 7$?

$\boxed{\text{A: } \int_2^4 2\pi(7 - x)(q(x) - p(x))dx}$

B: $\int_2^4 2\pi x(q(x) - p(x))dx$

C: $\int_2^4 2\pi(7 + x)(q(x) - p(x))dx$

D: $\int_2^4 2\pi(x - 7)(q(x) - p(x))dx$

11. [5 points] A giant table leg is being built by rotating the region bounded by the graph of $y = \frac{1}{2}\cos(2\pi x) + 2$, the x-axis, the line $x = 0$, and the line $x = 1$ about the x-axis. Assume the units of x and y are in meters. Write an integral which gives the volume of the table leg. Do not evaluate the integral. What are the units of this integral?

The volume of the table leg is given by the integral $\int_0^1 \pi(\frac{1}{2}\cos(2\pi x)+2)^2 dx$

The units of this integral are m^3