Math 116 — Second Midterm

March 24th, 2014

Name: EXAM SOLUTIONS

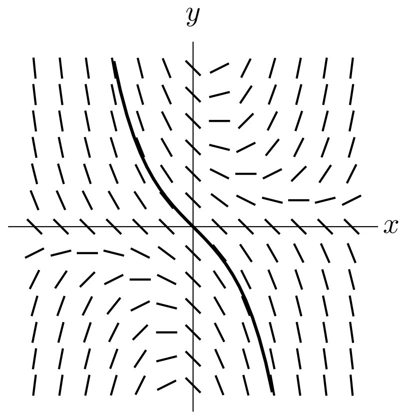
Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You may use a calculator to evaluate any integral unless specifically instructed otherwise. If you use a calculator to evaluate an integral, write the integral you are evaluating on your exam and indicate that you found the answer with a calculator.

Problem	Points	Score
1	10	
2	11	
3	11	
4	12	
5	7	
6	6	
7	7	
8	4	
9	10	
10	12	
11	10	
Total	100	

1. [10 points] Consider the differential equation $y' = xy - 1$.

a. [2 points] The slope field of $y' = xy - 1$ is shown below. On the graph, sketch a solution curve passing through the point $(0, 0)$.



b. [5 points] Starting with the initial condition $y(0) = 0$, use Euler's method with 3 steps to estimate $y(3/2)$. Show your work for each step.

Solution:
$y(1/2) \approx y(0) + \frac{1}{2}y'(0) = -\frac{1}{2}$
$y(1) \approx y(1/2) + \frac{1}{2}y'(1/2) \approx -9/8$
$y(3/2) \approx y(1) + \frac{1}{2}y'(1) \approx -35/16.$

c. [3 points] Can you determine if your estimate of $y(3/2)$ is an underestimate or overestimate? Circle your answer and **explain** your reasoning in one sentence.

Underestimate $\boxed{\textbf{Overestimate}}$ **Not enough information**

Solution:
y is concave down on the interval $(0, \infty)$ therefore Euler's method will give an overestimate.

2. [11 points] Abby and Brenda are alpacas running around in the xy-plane. Abby's position t minutes after she starts running is $(\cos(\pi t), 1)$ and Brenda's position t minutes after she starts running is $(\frac{t}{2}, e^{1-(t/2)^2})$. Both alpacas begin running at the same time.

a. [3 points] Do Brenda and Abby ever collide? If so at what time(s) does this occur?

Solution:
For Abby and Brenda to collide we must solve the equations $1 = e^{1-(t/2)^2}$ and $\frac{t}{2} = \cos(\pi t)$. For the first equation we must have $1 - (t/2)^2 = 0$ therefore $t = \pm 2$. Negative time doesn't make sense in this probelm so we only take $t = 2$.
Plugging 2 into the second equation both sides are equal. So Abby and Brenda collide when $t = 2$.

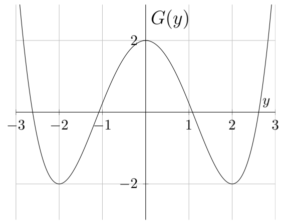
b. [5 points] Does Brenda or Abby ever stop moving at any time in the interval $[2.5, 4.5]$? If so, which alpaca stops and at what time(s) does this occur?

Solution:
$A'(t) = (-\pi \sin(\pi t), 0)$ so Abby stops moving whenever $\sin(\pi t) = 0$ so whenever t is an integer. Thus Abby stops when $t = 3$ or 4.
$B'(t) = (\frac{1}{2}, te^{1-(t/2)^2})$ the first coordinate can never be zero so Brenda is always moving.

c. [3 points] Write an integral which gives the distance traveled by Brenda in the first 5 minutes she is running. Please circle your answer.

Solution:
Using the parametric arc length formula we get $\displaystyle\int_0^5 (\frac{1}{4} + t^2 e^{2-2(t/2)^2})^{1/2}\, dt$.

3. [11 points] The graph of $G(y)$ is shown below. Suppose that $G'(y) = g(y)$. Consider the differential equation $\dfrac{dy}{dt} = g(y)$.



Note again that $\frac{dy}{dt} = g(y)$ and the given graph depicts $G(y)$ **not** $g(y)$.

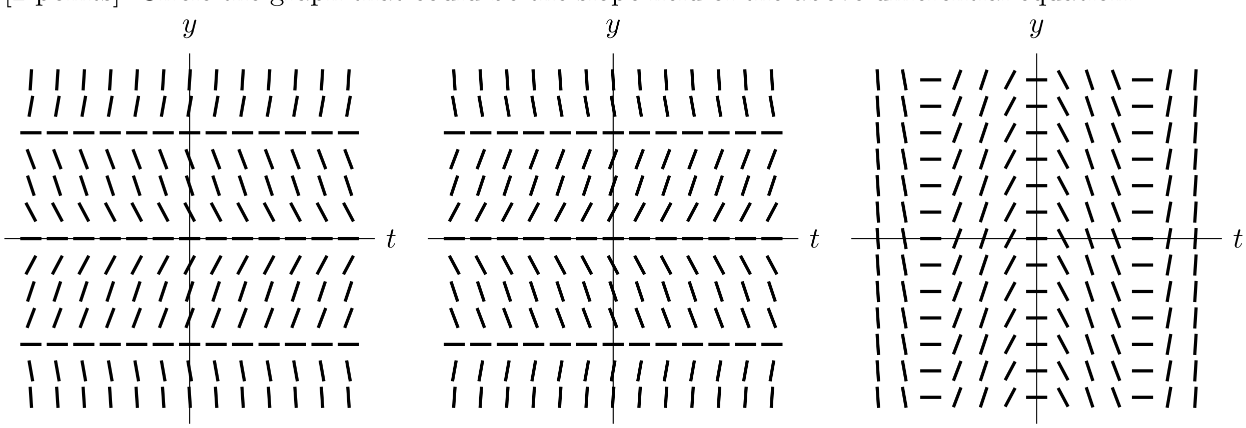
a. [6 points] The differential equation has 3 equilibrium solutions. Find the 3 solutions and indicate whether they are stable or unstable by circling the correct answer.

Equilibrium solution 1: -2 **Stable** $\boxed{\textbf{Unstable}}$

Equilibrium solution 2: 0 $\boxed{\textbf{Stable}}$ **Unstable**

Equilibrium solution 3: 2 **Stable** $\boxed{\textbf{Unstable}}$

b. [2 points] Circle the graph that could be the slope field of the above differential equation.



$\boxed{\text{I}}$ II III

c. [3 points] Suppose $y_1(t), y_2(t)$ and $y_3(t)$ are all solutions of the differential equation with different initial conditions as indicated below:

- $y_1(t)$ solves the differential equation with initial condition $y(0) = -2$.
- $y_2(t)$ solves the differential equation with initial condition $y(0) = 1.5$.
- $y_3(t)$ solves the differential equation with initial condition $y(0) = -2.1$.

Compute the following limits:

$\lim\limits_{t\to\infty} y_1(t) = -2$ $\lim\limits_{t\to\infty} y_2(t) = 0$ $\lim\limits_{t\to\infty} y_3(t) = -\infty$ or DNE

4. [12 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

a. [2 points] The differential equation $y' = \sin(\sin(y))$ has an infinite number of equilibrium solutions.

$\boxed{\text{True}}$ False

b. [2 points] If $C(x)$ is a cumulative distribution function then $\int_{-\infty}^{\infty} C(x)\,dx$ converges.

True $\boxed{\text{False}}$

c. [2 points] The integral $\int_0^1 \frac{1}{\sin(x)}\,dx$ converges

True $\boxed{\text{False}}$

d. [2 points] If $p(x)$ is a probability density function with $p(5) = 0$ then 5 cannot be the mean of the probability distribution.

True $\boxed{\text{False}}$

e. [2 points] If c is any constant then $y = 1 + ce^{-\frac{1}{2}x^2}$ is a solution to the differential equation $y' = x - xy$.

$\boxed{\text{True}}$ False

f. [2 points] The area of the region enclosed by the graph of $r = 2\sin(\theta)$ in the cartesian plane is given by $\displaystyle\int_0^{2\pi} 2\sin(\theta)^2\,d\theta$

True $\boxed{\text{False}}$

5. [7 points] The Terrible Telemarketing corporation has realized that people often hang up on their telemarketing calls. After collecting data they found that the probability that someone will hang up the phone at time t seconds after the call begins is given by the probability density function $p(t)$. The formula for $p(t)$ is given below.

$$p(t) = \begin{cases} 0 & t < 0 \\ te^{-ct^2} & t \geq 0 \end{cases}$$

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a. [5 points] Find the value of c so that $p(t)$ is a probability density function.

Solution:
We must have $\int_0^\infty te^{-ct^2}\,dt = 1$. Let $u = ct^2$ then $du = 2ctdt$. Thus we get the integral $\frac{1}{2c}\int_0^\infty e^{-u}du = \frac{1}{2c}\lim_{N\to\infty} -e^{-u}|_0^N = \frac{1}{2c}$. Thus we have $1 = \frac{1}{2c}$ so $c = \frac{1}{2}$.

b. [2 points] What is the probability that someone will stay on the phone with a telemarketer for more than 4 seconds?

Solution:
The probability is $\int_4^\infty te^{-\frac{1}{2}t^2}\,dt = e^{-8}$.

6. [6 points] Consider the probability density function $q(t)$ shown below.

$$q(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{2} & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

a. [4 points] What is the cumulative distribution function $Q(t)$ of the density given by $q(t)$? Write your final answer in the answer blanks provided.

Solution:
$Q(t) = \int_{-\infty}^t q(s)\,ds = \int_0^t q(s)\,ds$. If $t < 0$ then $Q(t) = 0$ and if $0 \leq t < 2$ then $Q(t) = \int_0^t q(s)\,ds = \int_0^t \frac{s}{2}\,ds = t^2/4$. Then it follows that $Q(t) = 1$ for $t \geq 2$.

If $t < 0$ then $Q(t) = 0$

If $0 \leq t < 2$ then $Q(t) = t^2/4$

If $t \geq 2$ then $Q(t) = 1$

b. [2 points] What is the median of the distribution?

Solution: The median is the number T such that $Q(T) = \frac{1}{2}$. Thus we want $T^2/4 = \frac{1}{2}$. Therefore $T = \sqrt{2}$.

7. [7 points] Bill has just built a brand new $90,000$ L swimming pool. Bill is allergic to chlorine so instead he is using a filtration system to prevent algae from building up in the pool. Algae grows in the pool at a constant rate of 600 kg/day. The filtration system receives a constant supply of $70,000$ L/day of water and returns the water to the pool with 6/7ths of the algae removed. Let $A(t)$ be the amount of algae in the pool in kilograms t days after Bill has filled the pool with fresh (algae free) water.

a. [5 points] Write down the differential equation satisfied by $A(t)$. Include the initial condition.

Solution:
$\frac{dA}{dt}$ = Rate in-Rate out. Rate in = 600 kg/day. Rate out=flow rate$\times$ concentration$\times$ fraction removed$= 70,000 \times \frac{A}{90,000} \times 6/7 = \frac{2}{3}A$. Thus $\frac{dA}{dt} = 600 - \frac{2}{3}A$.

$$\frac{dA}{dt} = 600 - \frac{2}{3}A$$

Initial condition: $A(0) = 0$

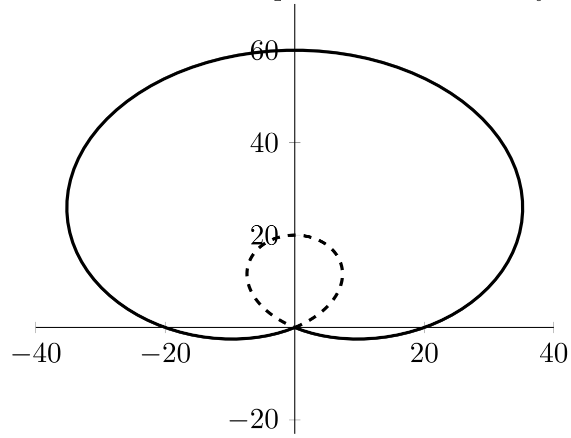
b. [2 points] Find all the equilibrium solutions of the differential equation.

Solution:
We want to solve $\frac{dA}{dt} = 600 - \frac{2}{3}A = 0$. Therefore $A = 900$ is the only equilibrium solution.

8. [4 points] Consider the differential equation $y' = e^y$. Solve the differential equation with initial condition $y(0) = 1$.

Solution:
The equation $\frac{dy}{dx} = e^y$ is separable so we have $e^{-y}dy = dx$. Integrating both sides we get $-e^{-y} = x + c$. Solving the equation we get $y = -\ln(c - x)$. To solve for c we take $y(0) = -\ln(c) = 1$. Therefore $c = \frac{1}{e}$. So the solution is $y = -\ln(\frac{1}{e} - x)$.

9. [10 points] Linda is designing a pond with a flat rock at one end. The rock plus the pond are in the shape of a cardioid. Plans for her pond design are depicted below. The cardioid has equation $r = 20 + 40\sin\theta$ where r is in feet and θ is in radians. The inner loop of the cardioid forms the shape of the rock and the outer loop forms the boundary of the pond.



a. [2 points] Find all values of θ between 0 and 2π for which $r = 0$.

Solution: First we set $r = 20 + 40\sin\theta = 0$. Therefore rearranging $\sin\theta = -\frac{1}{2}$. The solutions between 0 and 2π are $\theta = 7\pi/6, 11\pi/6$.

b. [4 points] Write an integral or sum of integrals which give(s) the perimeter of the boundary of the pond. Note this is the perimeter of the part of the cardioid drawn with a solid line.

Solution:
We will need to use the polar arc length formula so we need to calculate $r' = 40\cos\theta$. The arc length can be written as a single integral $\int_{-\pi/6}^{7\pi/6} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta$. Writing the arc length as two integrals we get $\int_0^{7\pi/6} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta + \int_{11\pi/6}^{2\pi} \sqrt{(40\cos\theta)^2 + (20 + 40\sin\theta)^2}\, d\theta$

c. [4 points] Write an integral or sum of integrals which give(s) the area of the top of the rock. Note this is the area enclosed by the dashed part of the cardioid.

Solution:
Now we need to use the polar area formula $\int_{7\pi/6}^{11\pi/6} \frac{1}{2}(20 + 40\sin\theta)^2\, d\theta$.

10. [12 points] Suppose that $g(x)$ and $h(x)$ are positive continuous functions on the interval $(0, \infty)$ with the following properties:

- $\int_1^\infty g(x)\, dx$ converges.
- $\int_0^1 g(x)\, dx$ diverges.
- $e^{-x} \leq h(x) \leq \frac{1}{x}$ for all x in $(0, \infty)$.

For each of the following questions, circle the correct answer.

a. [2 points] Does the integral $\displaystyle\int_1^\infty h(x)^2\, dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

b. [2 points] Does the integral $\displaystyle\int_0^1 h(x)\, dx$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

c. [2 points] Does the integral $\displaystyle\int_1^\infty h(1/x)\, dx$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

d. [2 points] Does the integral $\displaystyle\int_0^1 g(x)h(x)\, dx$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

e. [2 points] Does the integral $\displaystyle\int_1^\infty g(x)h(x)\, dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

f. [2 points] Does the integral $\displaystyle\int_1^\infty e^x g(e^x)\, dx$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

11. [10 points]

a. [5 points] Compute the improper integral $\displaystyle\int_0^1 \ln(x)\,dx$. Show your work.

Solution:
$\int_0^1 \ln(x)\,dx = \lim_{a\to 0^+} \int_a^1 \ln(x)\,dx = \lim_{a\to 0^+} x\ln(x)|_a^1 - \int_a^1 1\,dx = \lim_{a\to 0^+} -a\ln(a) - 1 + a$. Using either L'hopital's rule or the fact that polynomials dominate logarithms we have $\lim_{a\to 0^+} a\ln(a) = 0$. Therefore the integral is equal to -1.

b. [5 points] Use comparison of improper integrals to determine if the improper integral $\displaystyle\int_1^\infty \frac{\sin(x)+3}{x^2+2}$ converges or diverges. Show your work.

Solution:
We have the inequalities $\sin(x) + 3 \le 4$ and $\frac{1}{x^2+2} \le \frac{1}{x^2}$. Therefore $\int_1^\infty \frac{\sin(x)+3}{x^2+1}\,dx \le \int_1^\infty \frac{4}{x^2}\,dx = 4\int_1^\infty \frac{1}{x^2}\,dx$. This integral is a p-integral with $p = 2 > 1$ so it converges. Therefore $\int_1^\infty \frac{\sin(x)+3}{x^2+2}\,dx$ converges by comparison.