Math 116 — Final Exam

April 28, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You may use a calculator to evaluate any integral unless specifically instructed otherwise. If you use a calculator to evaluate an integral, write the integral you are evaluating on your exam and indicate that you found the answer with a calculator.
10. On the last page of this exam you will find a page containing formulas for some common Taylor series.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	12	
6	8	
7	8	
8	12	
9	10	
10	10	
Total	100	

1. [10 points]

a. [4 points] Consider the differential equation $y' = y^2 + y - 2$. Find all of the equilibrium solutions of the differential equation and indicate whether they are stable or unstable. Circle your answers.

Solution: $y' = y^2 + y - 2 = (y+2)(y-1)$. Therefore the equlibrium solutions are $y = -2$ which is stable and $y = 1$ which is unstable.

b. [4 points] Solve the differential equation $y' = y^2$ with initial condition $y(0) = 1$.

Solution: Separating variables we have $\frac{dy}{y^2} = dx$. Integrating both sides we have $-\frac{1}{y} = x + c$ therefore $y = \frac{-1}{x+c}$. Plugging in the initial condition we must have $c = -1$. So $y = \frac{1}{1-x}$.

c. [2 points] Which of the following functions is a solution to the differential equation $y' = \sin(x) + y$? Circle your answer.

$y = \frac{1}{2}(\sin(x) + \cos(x))$

$y = -\frac{1}{2}(\sin(x) - \cos(x))$

$y = \frac{1}{2}(\sin(x) - \cos(x))$

$\boxed{y = -\frac{1}{2}(\sin(x) + \cos(x))}$

2. [10 points] Consider an outdoor pool initially filled with 20,000 gallons of water. Each day 4% of the water in the pool evaporates. Each morning at 10:00am, W gallons of water are added back to the pool where W is a constant.

a. [3 points] Let A_n be the number of gallons of water in the pool immediately after water is added back to the pool for the n^{th} time. Given that $A_1 = 19200 + W$, find A_2 and A_3. Put your final answers in the answer blanks.

Solution:
$A_2 = (20,000)(\frac{24}{25})^2 + W(\frac{24}{25}) + W.$
$A_3 = (20,000)(\frac{24}{25})^3 + W(\frac{24}{25})^2 + W(\frac{24}{25}) + W.$

b. [4 points] Find a closed form expression for A_n (i.e. evaluate any sums and solve any recursion). Note your answer may contain the constant W.

Solution: $A_n = \frac{24}{25}A_{n-1} + W$. Expanding this recursion or following the pattern from part a we have $A_n = 20,000(\frac{24}{25})^n + \sum_{k=0}^{n-1} W(\frac{24}{25})^k$. Using the formula for finite geometric series we have $A_n = 20,000(\frac{24}{25})^n + 25W(1 - (\frac{24}{25})^n)$.

c. [3 points] If the pool has a maximum capacity of 25,000 gallons, find the largest value of W so that the pool does not overflow eventually.

Solution: Depending on the value of W, A_n is always increasing or always decreasing. Therefore the amount of water in the pool is the largest either when it is first filled at 20,000 gallons or when n approaches infinity where we have $\lim_{n\to\infty} A_n = 25W$. Therefore our only restriction is $25W \leq 25,000$ thus $W \leq 1,000$. So the largest possible value is $W = 1,000$.

3. [10 points] For each of the following questions circle the correct answer.

a. [2 points] What is the value of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{n!}$?

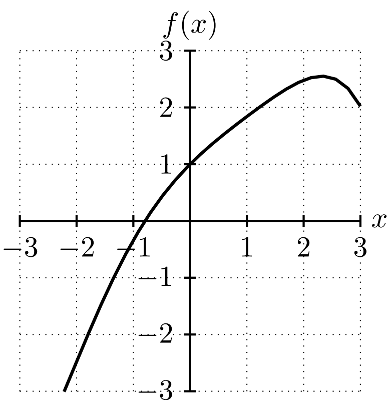
$\cos(2)$ $\qquad$ e^{-2} $\qquad$ $\cos(4)$ $\qquad$ $\boxed{e^{-4}}$

b. [2 points] What is the value of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}(-1)^n}{(2n+1)!}$?

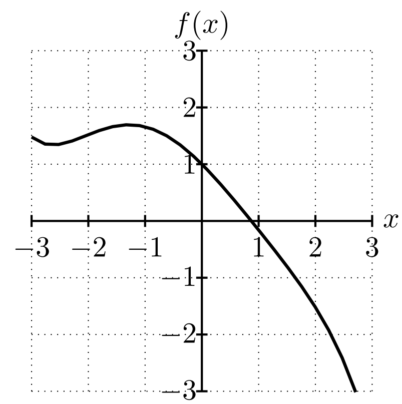
$\frac{1}{2}\sin(2)$ $\qquad$ $\sin(2) - 2$ $\qquad$ $\sin(2)$ $\qquad$ $\boxed{\frac{1}{2}(\sin(2) - 2)}$

c. [2 points] Suppose that $1 + x - \frac{1}{4}x^2 + \frac{1}{10}x^3$ is the 3rd degree Taylor polynomial for a function $f(x)$. Which of the following pictures could be a graph of $f(x)$?

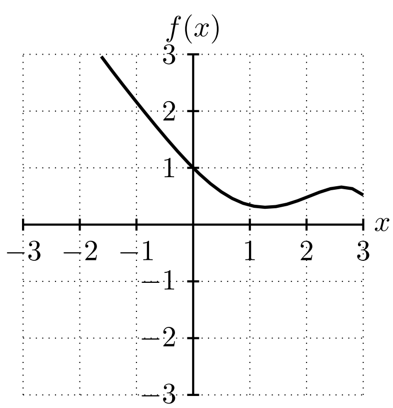
$\boxed{\text{I}}$



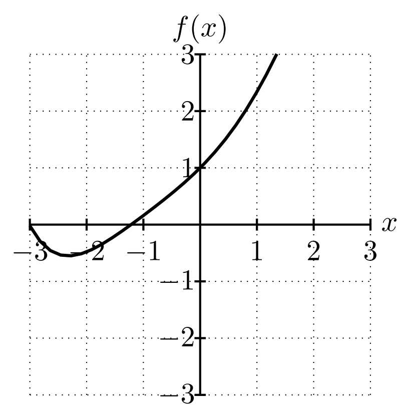
II



III



IV



d. [2 points] What is the Taylor series of $2xe^{x^2}$ centered at $x = 0$?

$\boxed{\displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n+1}}{n!}}$ $\qquad$ $\displaystyle\sum_{n=1}^{\infty} \frac{2x^{2n-1}}{n!}$ $\qquad$ $\displaystyle\sum_{n=1}^{\infty} \frac{2x^{2n+1}}{(n-1)!}$ $\qquad$ $\displaystyle\sum_{n=0}^{\infty} \frac{2x^{2n-1}}{n!}$

e. [2 points] The radius of convergence of the Taylor series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n 5^{-n}}{n+5}$ is $R = 5$. What is the interval of convergence of the series?

$\boxed{[-10, 0)}$ $\qquad$ $(-10, 0)$ $\qquad$ $(0, 10]$ $\qquad$ $[-10, 0]$ $\qquad$ $[0, 10)$

4. [10 points] Determine whether the following series converge or diverge. Show all of your work and justify your answer.

a. [5 points] $\displaystyle\sum_{n=1}^{\infty} \frac{8^n + 10^n}{9^n}$

Solution: $\lim_{n\to\infty} \frac{8^n+10^n}{9^n} = \infty$ therefore by the n^{th} term test the series diverges.

b. [5 points] $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^3 + n^2\cos(n)}$

Solution: $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n^3 + n^2\cos(n)} \leq \sum_{n=4}^{\infty} \frac{1}{n^2(n-1)} \leq \sum_{n=4}^{\infty} \frac{1}{n^2}$. The final series is a convergent p series since $p = 2 > 1$. Therefore the original series converges by comparison.

5. [12 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

a. [2 points] Suppose that an object has constant density δ and center of mass $(\bar{x}, \bar{y}, \bar{z})$. If the density of the object is doubled to 2δ then the center of mass changes to $(2\bar{x}, 2\bar{y}, 2\bar{z})$.

True $\boxed{\text{False}}$

b. [2 points] Every solution of the differential equation $y' = y$ is increasing.

True $\boxed{\text{False}}$

c. [2 points] If $f(x)$ is a continuous function and $F(x)$ is an antiderivative of $f(x)$, then $F(x) = \int_3^x f(t)dt + K$ for some constant K.

$\boxed{\text{True}}$ False

d. [2 points] If $g(x) = \displaystyle\int_{-e^x}^{e^x} t^2\,dt$ and $h(x) = \int_0^{2x} e^{t^2} dt$ then $g'(x) \le h'(x)$ for all $x > 1$.

$\boxed{\text{True}}$ False

e. [2 points] If $w(x)$ is a positive continuous function and the series $\sum_{n=1}^{\infty} w(n)$ converges then the integral $\int_1^{\infty} w(x)\,dx$ must also converge.

True $\boxed{\text{False}}$

f. [2 points] Suppose that a_n is a decreasing sequence and $0 \le a_n \le 1$ then $b_n = \cos(a_n)$ is a convergent sequence.

$\boxed{\text{True}}$ False

6. [8 points]

a. [2 points] Find all values of p for which the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^p}\,dx$ converges. You do not need to show your work. Circle your final answer.

Solution: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^p}\,dx = \int_{0}^{\infty} \frac{2}{(x^2+4)^p}\,dx \le \int_{1}^{\infty} \frac{2}{x^{2p}}/,dx$ thus $p > 1/2$.

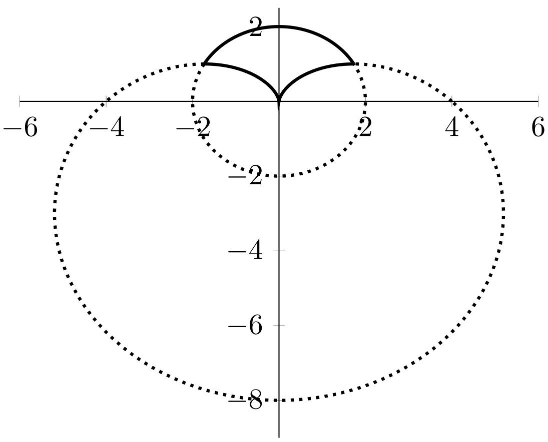
b. [2 points] Find all values of p for which the integral $\displaystyle\int_{e}^{\infty} \frac{e^{px}}{x^3}\,dx$ converges. You do not need to show your work. Circle your final answer.

Solution:
If $p \le 0$ then $\displaystyle\int_{e}^{\infty} \frac{e^{px}}{x^3}\,dx \le \int_{e}^{\infty} \frac{1}{x^3}\,dx$ which converges thus the integral converges if $p \le 0$. If $p > 0$ then $\lim_{x\to\infty} frace^{px}x^3 = \infty$ so the integral will diverge. Therefore the answer is $p \le 0$.

c. [4 points] Find the **radius** of convergence of the Taylor series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n}}{n2^n}$.

Solution: Using the ratio test we consider $\lim_{n\to\infty} |\frac{x^{2(n+1)}n2^n}{x^{2n}(n+1)2^{n+1}}| = \lim_{n\to\infty} |\frac{x^2 n}{2(n+1)}| = |x^2/2|$. In order for the series to converge we must have $|x^2/2| < 1$. Therefore $|x| < \sqrt{2}$. So the radius of converge is $R = \sqrt{2}$.

7. [8 points] Roy the alpaca is designing a pool and a deck for his family. The pool has the shape of a cardioid whose equation is given by $r = 4 - 4\sin(\theta)$ where r is in meters and θ is a number between 0 and 2π. The deck will be built in the region that lies inside the circle $x^2 + y^2 = 4$ and outside the cardioid. The deck is depicted in the figure as the region enclosed by the solid lines



a. [1 point] Write the equation for the circle $x^2 + y^2 = 4$ in polar coordinates.

Solution: $r^2 = 4$ so $r = 2$

b. [2 points] Find the values of θ between 0 and 2π where the cardioid and the circle intersect.

Solution: Setting the two equations equal to each other we have $2 = 4 - 4\sin(\theta)$ thus $\sin(\theta) = \frac{1}{2}$. Therefore $\theta = \pi/6, 5\pi/6$.

c. [5 points] Write an expression involving integrals that gives the area of the region where the deck will be built. Do not evaluate your expression.

Solution: $\int_{\pi/6}^{5\pi/6} 2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\,d\theta = 4\pi/3 - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(4 - 4\sin(\theta)^2\,d\theta$.

8. [12 points] Suppose a_n and b_n are sequences of positive numbers with the following properties.

- $\sum_{n=1}^{\infty} a_n$ converges.
- $\sum_{n=1}^{\infty} b_n$ diverges.
- $0 < b_n \leq M$ for some positive number M.

For each of the following questions, circle the correct answer. No justification is necessary.

a. [2 points] Does the series $\sum_{n=1}^{\infty} a_n b_n$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

b. [2 points] Does the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

c. [2 points] Does the series $\sum_{n=1}^{\infty} \sqrt{b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

d. [2 points] Does the series $\sum_{n=1}^{\infty} \sin(a_n)$ converge?

$\boxed{\text{Converge}}$ Diverge Cannot determine

e. [2 points] Does the series $\sum_{n=1}^{\infty} (a_n + b_n)^2$ converge?

Converge Diverge $\boxed{\text{Cannot determine}}$

f. [2 points] Does the series $\sum_{n=1}^{\infty} e^{-b_n}$ converge?

Converge $\boxed{\text{Diverge}}$ Cannot determine

9. [10 points] Jennifer is designing a doorknob. The shape of the doorknob is the solid formed by rotating the region bounded by $y = 2 - \cos(2x)$, $y = \frac{1}{4}$, $x = \frac{\pi}{2}$, and the y-axis about the x-axis. Assume the units of x and y are inches.

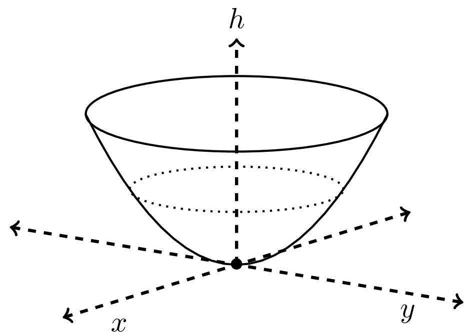
a. [5 points] Write an integral which gives the volume of the doorknob. Do not evaluate your integral. Circle your answer.

Solution: $\int_0^{\pi/2} \pi[(2 - \cos(2x))^2 - (1/4)^2]\, dx$

b. [5 points] The doorknob is to be made out of a material with constant density δ. The y-coordinate of the center of mass of the doorknob is $\bar{y} = 0$. Write an expression involving integrals which gives the x-coordinate of the center of mass of the doorknob. Do not evaluate your expression. Circle your answer.

Solution: $\dfrac{\int_0^{\pi/2} \pi x[(2-\cos(2x))^2-(1/4)^2]\, dx}{\int_0^{\pi/2} \pi[(2-\cos(2x))^2-(1/4)^2]\, dx}$

10. [10 points] Martin is having a party to celebrate the beginning of spring and he is serving punch out of a parabolic punch bowl. The bowl is sitting on a table (the xy-plane) as depicted in the figure below. At a height h above the table, the cross section of the bowl perpendicular to the h-axis is a circle with equation, $h = 4x^2 + 4y^2$. The punch bowl is 1 meter tall. Assume the units of x, y, and h are in meters and the density of the punch is 1200 kg/m^3. Recall the gravitational constant is $g = 9.8$ m/s^2.



a. [5 points] Write an expression for the mass of a slice of punch of thickness Δh meters at a height h meters above the table.

Solution: $M = 1200\pi\frac{h}{4}\Delta h$

b. [5 points] Assuming the bowl is filled with punch up to a height of $h = 1/2$, write an integral which gives the amount of work needed to lift all of the punch over the rim of the bowl. Do not evaluate your integral.

Solution: $\int_0^{1/2} 1200(9.8)\pi\frac{h}{4}(1-h)\,dh$

You may find the following expressions useful.

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\,x^2 + \frac{p(p-1)(p-2)}{3!}\,x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$