Math 116 — First Midterm

February 9, 2015

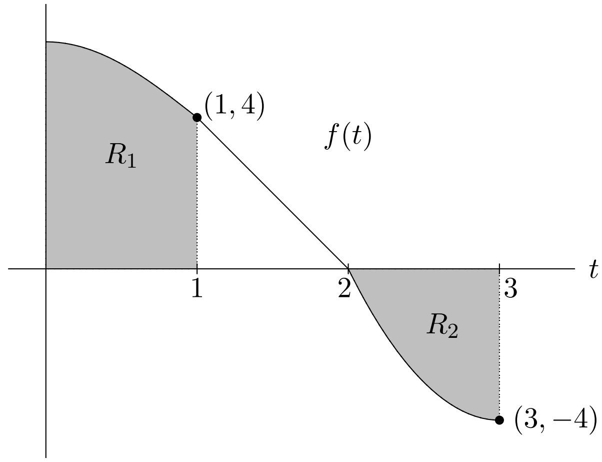
Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	13	
3	13	
4	14	
5	11	
6	10	
7	16	
8	9	
Total	100	

1. [14 points] While you are trying to fill your old bucket with water, it begins to leak. Suppose the continuous function $f(t)$ is the rate of change of the volume of water in the bucket, in gallons per minute, t minutes after it begins to leak. A graph of $f(t)$ for $0 \le t \le 3$ is shown below. The function $f(t)$ is linear for $1 \le t \le 2$. The region R_1 has area 5.8, and the region R_2 has area 3. There are 7 gallons of water in the bucket at $t = 1$.



a. [5 points] Write an expression involving integrals for $A(t)$, the volume of water in the bucket, in gallons, t minutes after the bucket began to leak where $0 \le t \le 3$. Your expression may contain the function f.

Solution: $A(t) = 7 + \int_1^t f(x)dx$

b. [2 points] How much water was in the bucket when it began to leak? How much water was in the bucket 3 minutes after it began to leak? Fill in the blanks below.

Solution:
There were 1.2 gallons of water in the bucket when it began to leak.
There were 6 gallons of water in the bucket 3 minutes after it began to leak.

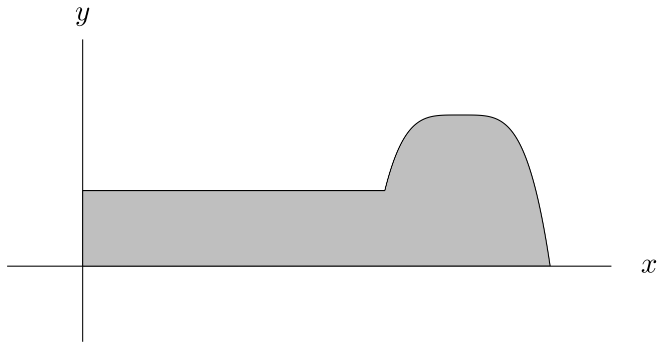
c. [3 points] Write an expression involving an integral for the average rate of change of the amount of water in the bucket during the first three minutes after it began to leak, and find the value of your expression, including units.

Solution: $\frac{1}{3-0}\int_0^3 f(t)dt = \frac{1}{3}[A(3) - A(0)] = \frac{4.8}{3}$ gal/min.

d. [4 points] For $t \geq 3$, suppose $f(t)$ is linear with slope 1, but is only defined until the time when the bucket is empty. For what value of t is the bucket empty? (Remember that f is continuous as specified above).

Solution: For $t \geq 3$, $f(t)$ is linear with slope 1 and passes through the point $(3, -4)$. So, $f(t) = t - 7$ for $t \geq 3$ until the time when the bucket is empty. To find the t value where the bucket is empty, we can solve $\displaystyle\int_3^t x - 7dx = -6$ to get $t^2 - 14t + 45 = 0$. We can factor the quadratic polynomial in t to get that $t = 5$ or $t = 9$. Then $f(t)$ will be defined until $t = 5$.

2. [13 points] Fred is designing a plastic bowl for his dog, Fido. Fred makes the bowl in the shape of a solid formed by rotating a region in the xy-plane around the y-axis. The region, shaded in the figure below, is bounded by the x-axis, the y-axis, the line $y = 1$ for $0 \le x \le 4$, and the curve $y = -(x-5)^4 + 2$ for $4 \le x \le 2^{1/4} + 5$. Assume the units of x and y are inches.



a. [7 points] Write an expression involving one or more integrals which gives the volume of plastic needed to make Fido's bowl. What are the units of your expression?

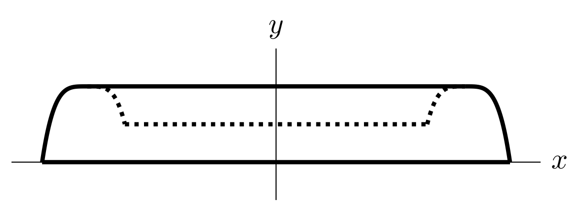
Solution: Using the cylindrical shell method, we have that the volume of plastic needed to make Fido's bowl is given by $\displaystyle\int_0^4 2\pi x dx + \int_4^{5+2^{\frac{1}{4}}} 2\pi x(2-(x-5)^4)dx$.

Using the washer method, we have that the volume of plastic needed to make Fido's bowl is given by $\displaystyle\pi\int_0^1 (5+(2-y)^{1/4})^2 dy + \pi\int_1^2 (5+(2-y)^{1/4})^2 - (5-(2-y)^{1/4})^2 dy$.

The units for either expression are in^3.

b. [6 points] Fred wants to wrap a ribbon around the bowl before he gives it to Fido as a gift. The figure below depicts the cross section of the bowl obtained by cutting it in half across its diameter. The thick solid curve is the ribbon running around this cross section, and the dotted curve is the outline of the cross section which is not in contact with the ribbon. Write an expression involving one or more integrals which gives the length of the thick solid curve in the figure (the length of ribbon Fred needs to wrap the bowl).

Solution: The length of ribbon Fred needs to wrap the bowl is given by
$\displaystyle 10 + 2(2^{\frac{1}{4}}+5) + 2\int_5^{5+2^{\frac{1}{4}}} \sqrt{1+16(x-5)^6}dx$.



3. [13 points] Use the table and the fact that

$$\int_0^{10} f(t)dt = 350$$

to evaluate the definite integrals below exactly (i.e., no decimal approximations). Assume $f'(t)$ is continuous and does not change sign between any consecutive t-values in the table.

t	0	10	20	30	40	50	60
$f(t)$	0	70	e^5	e^3	0	$\pi/2$	π

a. [4 points] $\displaystyle\int_0^{10} tf'(t)dt$

Solution:

$$\begin{aligned}\int_0^{10} tf'(t)dt &= tf(t)|_0^{10} - \int_0^{10} f(t)dt\\ &= 10f(10) - \int_0^{10} f(t)dt\\ &= 700 - 350\\ &= 350.\end{aligned}$$

b. [4 points] $\displaystyle\int_{20}^{30} \frac{f'(t)}{f(t)}dt$

Solution:

$$\begin{aligned}\int_{20}^{30} \frac{f'(t)}{f(t)}dt &= \int_{f(20)}^{f(30)} \frac{1}{u}du\\ &= \ln|u||_{f(20)}^{f(30)}\\ &= \ln|f(30)| - \ln|f(20)|\\ &= 3-5\\ &= -2.\end{aligned}$$

c. [5 points] $\displaystyle\int_{50}^{60} f(t)f'(t)\sin(f(t))dt$

Solution:

$$\begin{aligned}\int_{50}^{60} f(t)f'(t)\sin(f(t))dt &= \int_{f(50)}^{f(60)} w\sin(w)dw\\ &= -w\cos(w)|_{f(50)}^{f(60)} + \int_{f(50)}^{f(60)} \cos(w)dw\\ &= -\pi\cos(\pi) + \int_{\pi/2}^{\pi} \cos(w)dw\\ &= \pi - 1\end{aligned}$$

4. [14 points] The function

$$f(x) = \sin(\sqrt{x})$$

does not have an antiderivative that can be written in terms of elementary functions. However, we can use the second fundamental theorem of calculus to construct an antiderivative for f. We define an antiderivative F of f by

$$F(x) = \int_0^x \sin(\sqrt{t})dt.$$

a. [2 points] The concavity of F does not change on the interval $\left(0, \frac{\pi^2}{4}\right)$. Determine the concavity of F on $\left(0, \frac{\pi^2}{4}\right)$ and circle one of the options below. No justification is needed.

Solution:

$\boxed{\textbf{Concave Up}}$ **Concave Down** **Neither**

b. [2 points] Using the blanks provided, order from least to greatest

$$F\left(\frac{\pi^2}{4}\right), \quad \mathrm{LEFT}\,(100)\,, \quad \mathrm{RIGHT}\,(100)\,, \quad \mathrm{MID}\,(100)\,, \quad \mathrm{TRAP}\,(100)\,,$$

where all the approximations are of the definite integral given by $F\left(\frac{\pi^2}{4}\right)$. No justification is needed.

Solution: LEFT (100) $\leq$ TRAP (100) $\leq$ $F\left(\frac{\pi^2}{4}\right)$ $\leq$ MID (100) $\leq$ RIGHT (100)

c. [4 points] Write out, but do not compute, MID (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: $\mathrm{MID}\,(3) = \left[\sin\left(\sqrt{\frac{\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{24}}\right) + \sin\left(\sqrt{\frac{5\pi^2}{24}}\right)\right]\left(\frac{\pi^2}{12}\right)$

d. [4 points] Write out, but do not compute, TRAP (3) to approximate $F\left(\frac{\pi^2}{4}\right)$.

Solution: We have $\mathrm{LEFT}\,(3) = \left(\sin(0) + \sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$,
and $\mathrm{RIGHT}\,(3) = \left(\sin\left(\sqrt{\frac{\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{2\pi^2}{12}}\right) + \sin\left(\sqrt{\frac{3\pi^2}{12}}\right)\right)\left(\frac{\pi^2}{12}\right)$.
Then $\mathrm{TRAP}\,(3) = \frac{\mathrm{LEFT}(3)+\mathrm{RIGHT}(3)}{2}$.

e. [2 points] If you want to approximate $F\left(\frac{\pi^2}{4}\right)$ using right and left sums, what is the smallest number of subdivisions, n, you would have to use to guarantee that the difference between LEFT(n) and RIGHT(n) is less than or equal to 0.005?

Solution:

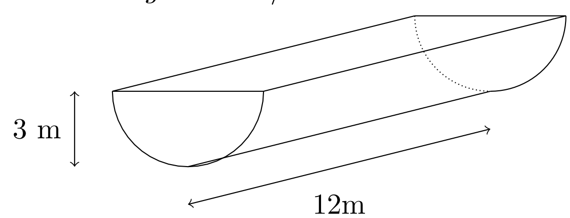
$$\begin{aligned} \text{RIGHT}\,(n) - \text{LEFT}\,(n) < 0.005 &\iff \left(\sin\left(\sqrt{\frac{\pi^2}{4}}\right) - \sin(0)\right)\frac{\frac{\pi^2}{4} - 0}{n} < 0.005 \\ &\iff n > \frac{\pi^2}{0.02} \approx 493. \end{aligned}$$

Since we need an integer number of subdivisions, we take $n = 494$.

5. [11 points] Robber baron and philanthropist Calvin Currency is making a large cash donation in \$100 bills. Before making the donation, he decides to fill an empty pool with the money. The pool is a half cylinder with radius 3 meters and length 12 meters as shown below. After an afternoon of diving into his pool of money and swimming around, the distribution of bills in the pool becomes nonuniform and so the density of money in the pool is given by

$$\delta(y) = 30,000\sqrt{\frac{10}{\pi}}e^{-y^2},$$

measured in bills per m^3, where y is height in meters measured from the bottom of the pool. Recall the gravitational constant is $g = 9.8$ m/s^2



a. [5 points] Write a definite integral which gives the volume of the pool.

Solution: The volume of the pool is $\displaystyle\int_0^3 12(2\sqrt{9-(y-3)^2})dy.$

b. [2 points] Write a definite integral which gives the value of the money in the pool, in dollars.

Solution: The value of money in the pool is given by $100\displaystyle\int_0^3 \delta(y)(12)(2\sqrt{9-(y-3)^2})dy.$

c. [4 points] Write a definite integral which gives the amount of work done in lifting the money out of the pool if each bill has mass 0.001 kg.

Solution: The work done in lifting the money out of the pool is given by
$\displaystyle\int_0^3 (0.001)\delta(y)(g)(12)(2\sqrt{9-(y-3)^2})(3-y)dy.$

6. [10 points] Your eccentric neighbor is rollerblading down the street away from you at constant speed with cardboard wings strapped to his back. He is 30 meters away from you when rockets strapped to his rollerskates ignite and quickly burn out. The **velocity** $v(t)$ of your neighbor, measured in meters per second, t seconds after he starts moving away from you is given below.

$$v(t) = \begin{cases} 4 & \text{if } 0 \le t \le 5 \\ -.64t^2 + 12.8t - 44 & \text{if } 5 < t \le 10 \\ -1.9t + 39 & \text{if } 10 < t \le 20 \end{cases}$$

a. [1 point] At what t value did the rockets ignite?

Solution:

The rockets ignited at $t =$ 5

Let $p(t)$ be the **distance** between you and your neighbor, measured in meters, t seconds after he starts moving away from you.

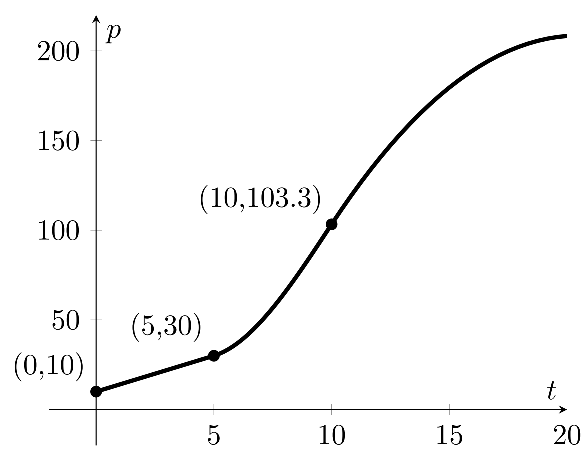
b. [4 points]
Determine the value of $p(0)$ and $p(10)$.

Solution:

$p(0) =$ 10
$p(10) =$ 310/3

c. [5 points] Sketch a well-labeled graph of $p(t)$ on the domain $0 \le t \le 20$ being sure the concavity of the graph is clear.

Solution:



7. [16 points] In each part, circle "True" if the statement is always true and circle "False" otherwise. No justification is necessary. Any unclear markings will be marked incorrect.

Solution:

a. [8 points] Suppose $g(x)$ is a positive function, defined for all real numbers x, with continuous first derivative.

(1) $\displaystyle\int_0^7 xg(x^2)\,dx = \int_0^7 g(u)\,du.$

True $\boxed{\text{False}}$

(2) $\displaystyle\int_0^7 xg(x^2)\,dx = \frac{1}{2}\int_0^{49} g(t)\,dt.$

$\boxed{\text{True}}$ False

(3) $\displaystyle\int_0^7 xg(x^2)\,dx = 7g(49) - \int_0^7 g(x^2)\,dx.$

True $\boxed{\text{False}}$

(4) $\displaystyle\int_0^7 xg(x^2)\,dx = \frac{49}{2}g(49) - \int_0^7 x^3g'(x^2)\,dx.$

$\boxed{\text{True}}$ False

b. [8 points] Suppose $h(y)$ is the density, in grams per cm, of a thin rod of length 10 cm, y cm from one end. Suppose the rod has mass M.

(1) $\displaystyle\int_0^5 h(y)\,dy = \frac{M}{2}.$

True $\boxed{\text{False}}$

(2) The center of mass of the rod is $\displaystyle\int_0^{10} yh(y)\,dy.$

True $\boxed{\text{False}}$

(3) If $h(y)$ is a constant function, then $h(y) = \frac{M}{10}$.

$\boxed{\text{True}}$ False

(4) The average value of $h(y)$ on $[0, 10]$ is $\frac{M}{10}$.

$\boxed{\text{True}}$ False

8. [9 points] Sally, the marine scientist, is reeling in a large shark she caught onto her boat. The edge of her boat lies 5 meters above the water as shown in the figure below. The total length of the sharking line is 30 meters. The shark weighs 500 newtons in water, and her sharking line weighs 30 newtons per meter out of water, and 10 newtons per meter in water. The figure below depicts this situation - the sharking line is the thick dark line and the boat is shaded. Write an expression which gives the work Sally does pulling the shark's snout to the surface of the water.

Solution: Suppose Sally has already reeled in x meters of line. The weight of the shark and remaining line is then $F(x) = 500 + 30(5) + 10(25 - x)$ N. Then the work done in pulling the shark's snout to the surface of the water is given by

$$\int_0^{25} F(x)dx = \int_0^{25} 900 - 10xdx = 19375J.$$