Math 116 — Final Exam

April 23, 2015

Name: EXAM SOLUTIONS

Instructor: Section:

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1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. On the last page of this exam you will find a page containing formulas for some common Taylor series.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	4	
3	5	
4	9	
5	10	
6	9	
7	9	
8	10	
9	10	
10	12	
11	12	
Total	100	

1. [10 points] Show that the following series converges. Also, determine whether the series converges conditionally or converges absolutely. Circle the appropriate answer below. **You must show all your work and indicate any theorems you use to show convergence and to determine the type of convergence**.

$$\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$$

[✓ CONVERGES CONDITIONALLY] CONVERGES ABSOLUTELY

Solution:
The series we obtain when we take the absolute value of the terms in the series above is $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. Now consider the integral $\int_2^{\infty} \frac{\ln(x)}{x} dx$. By making a change of variables we see that

$$\begin{aligned} \int_2^{\infty} \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_2^{b} \frac{\ln(x)}{x} dx \\ &= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} u du \\ &= \lim_{b\to\infty} \frac{(\ln(b))^2}{2} - \frac{(\ln(2))^2}{2} \\ &= +\infty \end{aligned}$$

and so the integral above diverges. Thus, the integral test implies that $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges.

Since $\frac{\ln(n+1)}{n+1} \leq \frac{\ln(n)}{n}$ and $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, we have that $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ converges by the alternating series test.

Altogether we have shown that the series $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is conditionally convergent.

2. [4 points]

a. [2 points] You are given that the power series $\sum_{n=0}^{\infty} C_n(x+3)^n$ converges when $x=-6$ and diverges when $x=1$. What are the largest and smallest possible values for the radius of convergence R?

Solution:

$$\underline{3} \le R \le \underline{4}$$

b. [2 points] Give the exact value of the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}}{n!} = -3 + 9 - \frac{27}{2} + \frac{81}{6} - \frac{243}{24} + \cdots$$

Solution: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}3^{n+1}}{n!} = -3e^{-3}$ using the Taylor series for xe^x centered at $x=0$.

3. [5 points] Determine whether the following integral converges or diverges. If the integral converges, circle the word "converges". If the integral diverges, circle "diverges". In either case, **you must show all your work and indicate any theorems you use**.

$$\int_0^1 \frac{\cos(x)}{x^2}\,dx$$

CONVERGES [✓ DIVERGES]

Solution: For $0 \le x \le 1$, we have that $\cos(x) \ge \cos(1)$. We also know that $\cos(1)\int_0^1 \frac{1}{x^2}dx$ diverges. Therefore, $\int_0^1 \frac{\cos(x)}{x^2}dx$ diverges by the comparison test.

4. [9 points] We can define the Bessel function of order one by its Taylor series about $x = 0$,

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

a. [3 points] Compute $J_1^{(2015)}(0)$. Write your answer in exact form and do not try evaluate using a calculator.

Solution: The 2015th derivative of J_1 at $x = 0$ corresponds to 2015! times the 1007th coefficient in the Taylor series above. This gives us that $J_1^{(2015)}(0) = \dfrac{-(2015)!}{(1007)!(1008)!2^{2015}}$.

$$J_1^{(2015)}(0) = \underline{\frac{-(2015)!}{(1007)!(1008)!2^{2015}}}$$

b. [4 points] Find $P_5(x)$, the Taylor polynomial of degree 5 that approximates $J_1(x)$ near $x = 0$.

Solution: $P_5(x) = \dfrac{x}{2} - \dfrac{x^3}{16} + \dfrac{x^5}{384}$

$$P_5(x) = \underline{\frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384}}$$

c. [2 points] Use the Taylor polynomial from the previous part to compute

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3}.$$

Solution: $\displaystyle\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \lim_{x\to 0} \frac{-\frac{x^3}{16} + \frac{x^5}{384}}{x^3} = -\frac{1}{16}$

$$\lim_{x\to 0} \frac{J_1(x) - \frac{1}{2}x}{x^3} = \underline{-\frac{1}{16}}$$

5. [10 points]

a. [5 points] Determine the **radius** of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^{4n}}{n^5 (16)^n}.$$

Solution: The above series will converge for x values such that

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} < 1$$

by the ratio test. We have

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{4n+4}}{(n+1)^5(16)^{n+1}}}{\frac{|x-5|^{4n}}{n^5(16)^n}} = \frac{1}{16}|x-5|^4.$$

and so the desired inequality holds if $\frac{1}{16}|x-5|^4 < 1$. This is equivalent to $|x-5| < 2$. Thus, the radius of convergence is 2.

The radius of convergence is $\underline{2}$.

b. [5 points] The power series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ has radius of convergence 1. Determine the **interval** of convergence for this power series.

Solution:

For $x = 1$, we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$. We have that $\displaystyle\lim_{n\to\infty} \frac{\frac{n+2}{n^4+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4+2n^3}{n^4+1} = 1$.

The limit comparison test then tells us that the series $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges if and only if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p > 1$ it converges, and so $\displaystyle\sum_{n=0}^{\infty} \frac{n+2}{n^4+1}$ converges.

For $x = -1$, we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+2)}{n^4+1}$. By the work shown above, this series converges absolutely.

The interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)x^n}{n^4+1}$ is then $-1 \le x \le 1$

The interval of convergence is $\underline{[-1,1]}$.

6. [9 points]

a. [3 points] Find the first three nonzero terms in the Taylor series for $\dfrac{1}{\sqrt{1-x^2}}$ centered at $x = 0$.

Solution: We have $(1+y)^{-1/2} = 1 - \frac{1}{2}y + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}y^2 + \cdots$. Substituting $y = -x^2$ gives us $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots$

b. [4 points] Use your answer from part (a) to find the first three nonzero terms in the Taylor series for $\arcsin(2x)$ centered at $x = 0$. Recall that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$.

Solution: Integrating the series from part (a) termwise gives us

$$\begin{aligned}\arcsin(x) &= \int 1dx + \int \frac{1}{2}x^2 dx + \int \frac{3}{8}x^4 dx + \cdots \\ &= C + x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots\end{aligned}$$

Since $\arcsin(0) = 0$, we must have $C = 0$. Then $\arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$, and from there a substitution gives us that $\arcsin(2x) = 2x + \frac{4}{3}x^3 + \frac{12}{5}x^5 + \cdots$.

c. [2 points] Find the values of x for which the Taylor series from part (b) converges.

Solution: The steps we took to get a Taylor series expansion for $\arcsin(x)$ do not change the radius of convergence. So, the Taylor series we found for $\arcsin(x)$ converges for $-1 < x < 1$. Then substituting $2x$ for x gives us that the series which gives our answer to (b) converges for $-1 < 2x < 1$, and so $-\frac{1}{2} < x < \frac{1}{2}$.

7. [9 points] Gwen lifts a bucket of sand straight up from the ground to a height of 10 meters at a constant speed of 0.5 meters per second. The sand is leaking out of the bucket at a rate of $r(t) = \frac{1}{t+1}$ kilograms per second, t seconds after she begins lifting. The bucket and the sand in the bucket together weigh 10 kg when she starts lifting. Recall the gravitational constant is $g = 9.8$ m/s^2.

a. [4 points] Suppose $M(x)$ is the mass of the bucket of sand (in kilograms) when she has lifted it x meters from the ground. Find an expression involving integrals for the work Gwen does lifting the bucket. Your answer can include the function M.

Solution: The work Gwen does lifting the bucket is

$$\int_0^{10} M(x) g dx.$$

b. [5 points] Find an expression, possibly involving integrals, for $M(x)$, the mass of the bucket of sand after Gwen has lifted it x meters.

Solution: The mass of the bucket after Gwen has lifted it x meters is

$$M(x) = 10 - \int_0^x \frac{2}{2s+1} ds.$$

8. [10 points] Consider the region A in the xy-plane bounded by $y = 1 - x^4$, the y-axis, and the x-axis in the first quadrant. The area of A is $\frac{4}{5}$.

a. [5 points] Suppose N is any positive whole number. Put the following quantities in order from least to greatest. MID(N), TRAP(N), RIGHT(N), LEFT(N), and the number $\frac{4}{5}$, where all of the approximations listed are for the integral $\int_0^1 (1 - x^4)\,dx$.

Solution:

RIGHT(N) $\leq$ **TRAP(N)** $\leq$ 4/5 $\leq$ **MID(N)** $\leq$ **LEFT(N)**

b. [5 points] Write an expression involving integrals that gives the volume of the solid formed by rotating the region A around the y-axis.

Solution: The volume of the solid formed by rotating the region A about the y-axis is $\int_0^1 2\pi x(1 - x^4)dx$ or equivalently $\int_0^1 \pi(1 - y)^{\frac{1}{2}}\,dy$

9. [10 points] Vic is planning to put ladybugs in his garden to eat harmful pests. The ladybug expert at the gardening store claims that the number of ladybugs in his garden can be modeled by the differential equation

$$\frac{dL}{dt} = \frac{L}{20} - \frac{L^2}{100}$$

where L is the number of ladybugs, in hundreds, in Vic's garden, t days after they are introduced.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate their stability.

Solution: To find the equilibrium solutions we set the right hand side of the equation above equal to 0. The resulting equation simplifies to $L(5 - L) = 0$. There is then an unstable equilibrium at $L = 0$ and a stable equilibrium at $L = 5$.

b. [2 points] If Vic starts his garden with 50 ladybugs, what will the long term population of ladybugs in his garden be according to the differential equation above?

Solution: The long term population is 500 ladybugs, as the solution to the differential equation above with initial condition $L(0) = .5$ will tend to the stable equilbrium at $L = 5$ as $t \to \infty$.

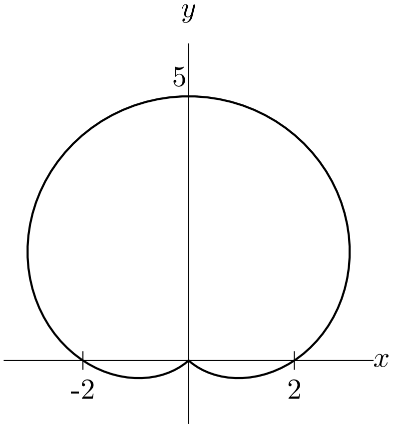
The long term population is **500 ladybugs**

c. [4 points] For what value of b is the function $L(t) = 5e^{bt}\left(4 + e^{bt}\right)^{-1}$ a solution to this differential equation.

Solution: We can compute that $\frac{dL}{dt} = 5be^{bt}\left(4 + e^{bt}\right)^{-1} - 5e^{2bt}\left(4 + e^{bt}\right)^{-2} = \frac{20be^{bt}}{(1 + e^{bt})^2}$ and $\frac{L}{20} - \frac{L^2}{100} = \frac{5e^{bt}\left(4 + e^{bt}\right)^{-1}}{20} - \frac{25e^{2bt}\left(4 + e^{bt}\right)^{-2}}{100} = \frac{e^{bt}}{(1 + e^{bt})^2}$. These two expressions will be equal provided that $b = \frac{1}{20}$.

$b =$ **1/20**

10. [12 points] Vic is watching the ladybugs run around in his garden. His garden is in the shape of the outer loop of a cardioid with polar equation $r = 2 + 3\sin\theta$ where r is measured in meters and θ is measured in radians. The outline of the garden is pictured below for your reference. At a time t minutes after he begins watching, Apple, his favorite red ladybug, is at the xy-coordinate $(\sin^2 t, \cos^2 t)$, and Emerald, his prized green ladybug, is at the xy-coordinate $(-\cos(2t), \sin(2t) + 1.5)$. Vic watches the ladybugs for 2π minutes.



Using the information above, circle the correct answer for each part below. There is only one correct answer for each part. You do not need to show your work.

a. [3 points] Which of the following integrals gives the area of the garden?

Solution:

A) $\frac{1}{2}\int_0^{\pi}(2+3\sin\theta)^2 d\theta$

B) $\frac{1}{2}\int_{\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})}(2+3\sin\theta)^2 d\theta$

[✓ **C)** $\frac{1}{2}\int_{-\arcsin(\frac{2}{3})}^{\pi+\arcsin(\frac{2}{3})}(2+3\sin\theta)^2 d\theta$]

D) $\frac{1}{2}\int_0^{2\pi}(2+3\sin\theta)^2 d\theta$

E) $\frac{1}{2}\int_{2\pi-\arcsin(\frac{2}{3})}^{4\pi-\arcsin(\frac{2}{3})}(2+3\sin\theta)^2 d\theta$

b. [3 points] Which of the following is **not** true about Apple while Vic is watching?

Solution:

A) Apple runs through the point $(\frac{1}{2}, \frac{1}{2})$ more than once.

B) Apple crosses the path made by Emerald exactly 4 times.

C) Apple's speed is zero at least once.

D) Apple does not leave the garden.

[✓ **E)** Apple is moving faster than Emerald for some of the time.]

c. [3 points] How far does Emerald run while Vic is watching?

Solution:

A) π meters

B) 2π meters

[✓ **C)** 4π meters]

D) $\sqrt{8}\pi$ meters

E) 8π meters

d. [3 points] After the 2π minutes, Vic stops watching. Apple runs from the point $(x, y) = (0, 1)$ in the positive y-direction with speed of $g(T) = 5Te^{-T}$, T seconds after Vic stops watching. Which of the following is true?

Solution:

[✓ **A)** Apple leaves the garden eventually, but never runs further than 5 meters total.]

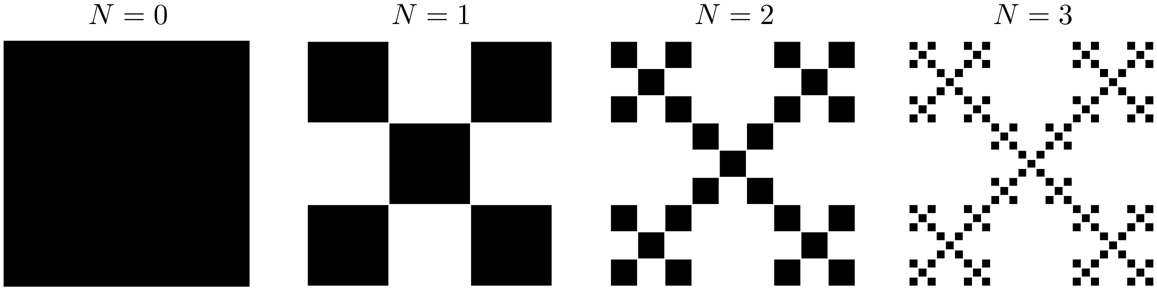
B) Apple's speed is always increasing after Vic stops watching.

C) If given enough time, Apple would eventually be more than 1000 meters from the garden.

D) Apple is still in the garden 5 minutes after Vic stops watching.

E) Apple changes direction, eventually.

11. [12 points] You construct a snowflake by starting with a square piece of paper of side length 3 inches. You divide the square into a three by three grid of squares of side length one and remove the four squares in the grid that share a side with the center square in the grid. For each remaining square in the grid, subdivide each of them into 9 equally sized squares and remove the four squares in each of these new grids that share a side with the center square in the grid. You continue in this manner for a long time.



a. [3 points] Write a formula that gives the perimeter, P_N, of the black squares that make up the snowflake after N steps.

Solution: $P_N = 12\left(\frac{5}{3}\right)^N$

b. [2 points] Find $\lim\limits_{N\to\infty} P_N$.

Solution: P_N tends to infinity as $N \to \infty$.

c. [3 points] Suppose $N \geq 1$. Write a sum that gives the area, A_N of all the squares you have **removed** after N steps.

Solution: $\sum\limits_{j=0}^{N-1} 4\left(\frac{5}{9}\right)^j$

d. [2 points] Write a closed form expression for A_N.

Solution: $A_N = 4\frac{1-\left(\frac{5}{9}\right)^N}{1-\frac{5}{9}}$

e. [2 points] Find the limit as $N \to \infty$ of your expression in (d).

Solution: $\lim\limits_{N\to\infty} A_N = 9$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$