Math 116 — Second Midterm

March 21, 2016

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	14	
2	13	
3	13	
4	5	
5	6	
6	15	
7	9	
8	13	
9	12	
Total	100	

1. [14 points] Determine if the following integrals converge or diverge. If the integral converges, circle the word "converges" and give the **exact value** (i.e. no decimal approximations). If the integral diverges, circle "diverges". In either case, **you must give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use**. Any direct evaluation of integrals must be done **without using a calculator**.

a. [7 points] $\displaystyle\int_1^\infty \frac{x}{e^{ax^2+1}}\,dx$, where $a > 0$ is a constant

Converges Diverges

Solution:

$$\int_1^\infty \frac{x}{e^{ax^2+1}}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{e^{ax^2+1}}\,dx = \lim_{b\to\infty}\int_{a+1}^{ab^2+1}\frac{1}{2ae^u}\,du =$$

$$= \lim_{b\to\infty}\left(\frac{-e^{-ab^2-1}}{2a} - \frac{-e^{-a-1}}{2a}\right) = \frac{e^{-a-1}}{2a} = \frac{1}{2ae^{a+1}}$$

In the second equality, we used the substitution $u = ax^2 + 1$.

b. [7 points] $\displaystyle\int_2^\infty \frac{x+\sin x}{x^2}\,dx$

Converges **Diverges**

Solution: Since $\sin x \geq -1$ for any x,

$$\frac{x + \sin x}{x^2} \geq \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2} \quad (*)$$

The improper integral

$$\int_2^\infty \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

diverges because the improper integral

$$\int_2^\infty \frac{1}{x}\, dx$$

diverges ($p = 1 \leq 1$) while the improper integral

$$\int_2^\infty \frac{1}{x^2}\, dx$$

converges ($p = 2 > 1$). So using $(*)$, the integral in question diverges by the comparison test.

Alternatively, we can use the inequality

$$\frac{x + \sin x}{x^2} \geq \frac{x-1}{x^2} \geq \frac{1}{2x}$$

which is valid for all $x \geq 2$, the p-test and the comparison test.

2. [13 points] Leia and Han are imprisoned in a cell whose door is made out of steel and has a thickness of 3 feet. Luke uses his lightsaber to cut through the door in the shape of the curve given by the polar coordinates equation

$$r = \frac{5}{3 + 2\cos\left(\theta + \frac{\pi}{4}\right)}$$

where r is measured in feet.

a. [6 points] Write an expression involving integrals for the volume of the piece that Luke cuts out of the door.

Solution:

$$3 \cdot \int_0^{2\pi} \frac{1}{2}\left(\frac{5}{3 + 2\cos\left(\theta + \frac{\pi}{4}\right)}\right)^2 d\theta \quad \mathrm{ft}^3$$

b. [7 points] Still considering the polar curve

$$r = \frac{5}{3 + 2\cos\left(\theta + \frac{\pi}{4}\right)}$$

graphed in the xy-plane, write an explicit expression involving integrals for the length of the **part** of the curve that lies **to the right** of the y-axis.

Solution:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{5}{3 + 2\cos\left(\theta + \frac{\pi}{4}\right)}\right)^2 + \left(\frac{10\sin\left(\theta + \frac{\pi}{4}\right)}{\left(3 + 2\cos\left(\theta + \frac{\pi}{4}\right)\right)^2}\right)^2}\, d\theta \quad \text{ft}$$

Alternatively:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{ft}$$

where

$$\frac{dx}{d\theta} = \frac{(-5\sin\theta)(3 + 2\cos(\theta + \pi/4)) + (5\cos\theta)2\sin(\theta + \pi/4)}{[3 + 2\cos(\theta + \pi/4)]^2}$$

and

$$\frac{dy}{d\theta} = \frac{(5\cos\theta)(3 + 2\cos(\theta + \pi/4)) + (5\sin\theta)2\sin(\theta + \pi/4)}{[3 + 2\cos(\theta + \pi/4)]^2}$$

3. [13 points] O-guk's playful son, O-ghan, is running on the xy-plane. His position t seconds after he begins running is

$$x = \sqrt{t} - 1 \qquad y = \sin(t) + 1.$$

Assume x and y are in meters.

a. [3 points] Does O-ghan pass though the origin? Briefly justify.

Solution: $x = 0$ when $\sqrt{t} - 1 = 0$ so when $t = 1$. For this value of t, $y = \sin(1) + 1 \neq 0$. So he didn't pass through the origin.

b. [4 points] How fast is O-ghan running at $t = 5$? Give your answer in **exact** form (i.e. no decimal approximations). Include **units**.

Solution:

$$\sqrt{\left(\frac{1}{2\sqrt{5}}\right)^2 + (\cos(5))^2} \qquad \frac{m}{s}$$

c. [6 points] Find an equation, in xy-coordinates, of the tangent line to his path at $t = 1$.

Solution: The slope of the tangent line is given by

$$m = \frac{dy/dt}{dx/dt} = \frac{\cos(1)}{\frac{1}{2\sqrt{1}}} = 2\cos(1)$$

The equation of the tangent line is $y - (\sin(1) + 1) = 2\cos(1)(x - 0)$ or equivalently

$$y = 2\cos(1)x + \sin(1) + 1$$

4. [5 points] Drake is running for president. Suppose $F(t)$ is the fraction of the total population of the country who supports him t months after he announces he is running. Drake gains supporters at a steady rate of 2% of the **total population** of the country per month, but he also steadily loses 3% of **his supporters** per month. Write a **differential equation** that models $F(t)$.

Solution:

$$\frac{dF}{dt} = 0.02 - 0.03F$$

5. [6 points] Adele is also running for president. Suppose $P(t)$, the total number of supporters she has in millions t days after she announces, is modeled by the differential equation

$$\frac{dP}{dt} = kP(100 - P)$$

with $k > 0$.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate stabilities for each. Make sure your answer is clear.

Solution: The equilibrium $P = 0$ is unstable and the equilibrium $P = 100$ is stable.

b. [2 points] If Adele starts with one million supporters, what is the maximum number of supporters she can get in the long run? You do not need to show your work.

Solution: $100,000,000$ supporters

6. [15 points] In the following questions, circle the correct answer. You do not need to show any work, but make sure your answer is clear. No points will be given for unclear answers.

a. [3 points] The value of A for which the function $y = e^{x^2+A^3x}$ solves the equation $y' + 8y = 2xy$ is

0 $\quad$ $\boxed{-2}$ $\quad$ -8 $\quad$ $-\sqrt{8}$ $\quad$ 1

b. [3 points] The function g is positive, decreasing and differentiable. The solution curves of the differential equation $y' = e^{-x}g(y)$ are

concave up $\quad$ $\boxed{\text{concave down}}$ $\quad$ changing concavity

c. [3 points] Suppose that $h(x)$ is an increasing differentiable function with $h(0) = 0$ and $\lim_{x\to\infty} h(x) = 5$. The value of the integral $\int_0^\infty (h(x))^4 h'(x)\,dx$

diverges $\quad$ $\boxed{\text{is } 5^4}$ $\quad$ is $5^4 - \frac{1}{5}$ $\quad$ is 1 $\quad$ is 0

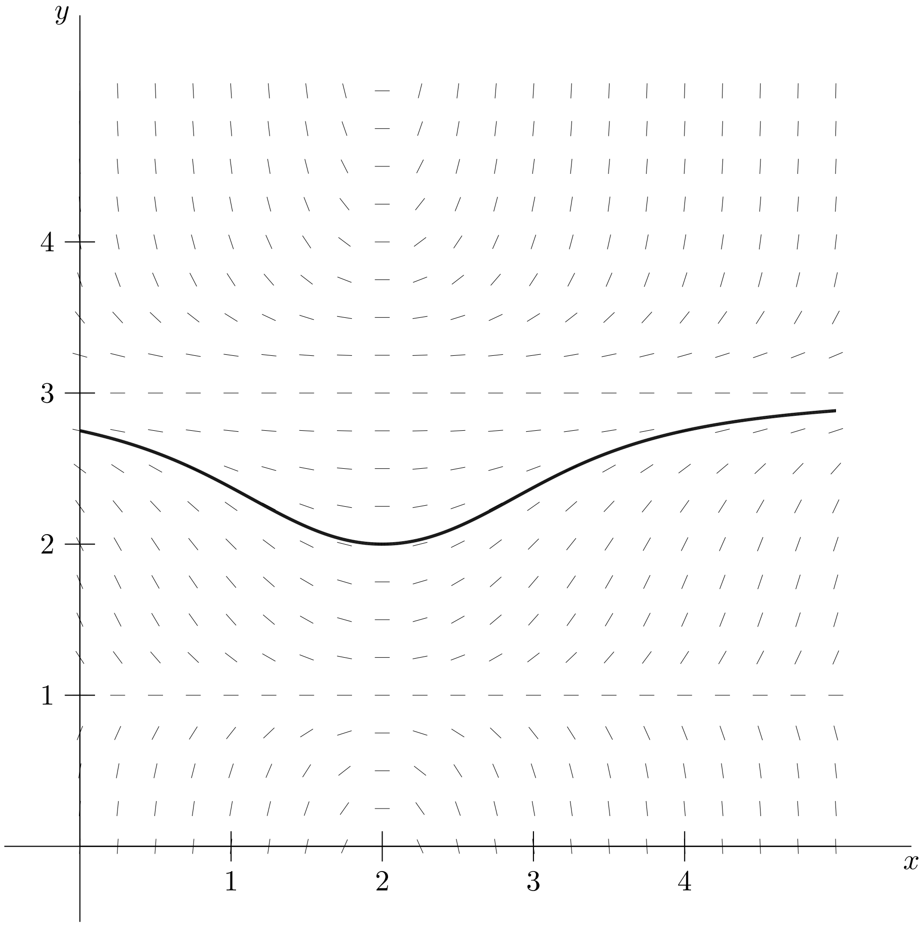
d. [3 points] Suppose $a \geq 1$ is a constant, and the function h satisfies $\frac{1}{x^{1/a}} \leq h(x) \leq \frac{1}{x^a}$ for $0 \leq x \leq 1$. The integral $\int_0^1 (h(x))^2 dx$ converges

always $\quad$ never $\quad$ $\boxed{\text{sometimes}}$

e. [3 points] The function f satisfies $\frac{1}{x^3} \leq f(x) \leq \frac{1}{x}$ for $x \geq 1$ and $f(x) = g(x^2)$. The integral $\int_1^\infty \frac{g(x)}{x} dx$ converges

$\boxed{\text{always}}$ $\quad$ never $\quad$ sometimes

7. [9 points] The graph of a slope field corresponding to a differential equation is shown below.



a. [3 points] On the slope field, carefully sketch a solution curve passing through the point $(2, 2)$ with domain $0 \leq x \leq 5$.

b. [4 points] The slope field pictured above is the slope field for one of the following differential equations. Which one? Circle your answer. You do not need to show your work.

$\boxed{\dfrac{dy}{dx} = (x-2)(y-1)(y-3)^2}$ $\qquad$ $\dfrac{dy}{dx} = (x+2)(y+1)(y+3)^2$

$\dfrac{dy}{dx} = (x-2)(y-1)^2(y-3)^2$ $\qquad$ $\dfrac{dy}{dx} = (x-2)(y-1)^2(y-3)$

c. [2 points] If we use Euler's method starting at the point $(2, 2)$ and use $\Delta x = 0.1$, would we get an overestimate or an underestimate for the value of $y(2.5)$? Circle your answer. You do not need to show your work.

overestimate $\qquad$ $\boxed{\text{underestimate}}$

8. [13 points] Brianne is hiking, and the temperature of the air in °C after she's traveled x km is a solution to the differential equation

$$y' + y\sin x = 0$$

a. [7 points] Find the general solution of the differential equation.

Solution: Writing the equation as $\frac{dy}{dx} = -y\sin x$ and separating the variables we get

$$\int \frac{1}{y}dy = \int -\sin x dx$$

$$\ln|y| = \cos x + C$$

$$y = Ae^{\cos x}$$

b. [2 points] If the temperature was 10 °C at the beginning of the hike, find $T(x)$, the temperature of the air in °C after she's traveled x km. Show your work.

Solution: From (a), $T(x) = Ae^{\cos x}$. Since $T(0) = 10$ we get $A = \frac{10}{e}$. Thus,

$$T(x) = \frac{10}{e}e^{\cos x}$$

c. [4 points] Brianne traveled 7 km on the hike. Using the information given in (b), find the coldest air temperature she encountered on the hike. Give an **exact** answer (i.e. no decimal approximations).

Solution: We want to find the minimum of the function $T(x) = \frac{10}{e}e^{\cos x}$ over the interval $[0, 7]$. The critical points are $0, \pi, 2\pi$. Checking the outputs of T at those points and the endpoint 7 we find that the minimum is $T(\pi) = \frac{10}{e^2}$.

9. [12 points]

a. [6 points] Show that the following integral diverges. Give full evidence supporting your answer, showing all your work and indicating any theorems about improper integrals you use.

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt$$

Solution: $t \geq 1 \Rightarrow \frac{1}{t} \leq 1 \Rightarrow \cos(\frac{1}{t}) \geq \cos(1)$ because the function $F(x) = \cos x$ is decreasing in the interval $[0, 1]$. Therefore,

$$\frac{\cos(\frac{1}{t})}{\sqrt{t}} \geq \frac{\cos(1)}{\sqrt{t}}$$

The improper integral

$$\int_1^\infty \frac{\cos(1)}{\sqrt{t}}\,dt = \cos(1)\int_1^\infty \frac{1}{\sqrt{t}}\,dt$$

diverges by the p-test since $p = \frac{1}{2} \leq 1$. So the integral

$$\int_1^\infty \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt$$

diverges by the comparison test (notice that $\cos(1) > 0$).

b. [6 points] Find the limit

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt}{\sqrt{x}}$$

Solution: Notice that by (a), this is $\frac{\infty}{\infty}$. We use L'Hopital's rule along with the 2nd Fundamental Theorem in the numerator:

$$\lim_{x\to\infty} \frac{\int_1^x \frac{\cos(\frac{1}{t})}{\sqrt{t}}\,dt}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{\cos(\frac{1}{x})}{\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} 2\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$$