Math 116 — Final Exam

April 21, 2016

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 14 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	8	
2	12	
3	12	
4	5	
5	6	
6	4	
7	6	
8	9	
9	6	
10	14	
11	5	
12	8	
Total	95	

1. [8 points] Suppose that $f(x)$ is a continuous function, and $F(x)$ is an antiderivative of $f(x)$. Assume that $\int_0^1 F(x)\,dx = 3$. A table of values for $F(x)$ is given below.

x	1	2	3	4	5
$F(x)$	1	-2	-4	3	1

Calculate the following quantities **exactly**. Show your work and do not write any decimal approximations.

a. [2 points] $\int_2^4 f(x)\,dx$

Solution: $\int_2^4 f(x)\,dx = F(4) - F(2) = 3 - (-2) = 5$ by the Fundamental Theorem of Calculus.

b. [2 points] The average value of f over the interval $[3, 5]$.

Solution: $\dfrac{\int_3^5 f(x)\,dx}{5-3} = \dfrac{F(5)-F(3)}{2} = \dfrac{1-(-4)}{2} = \dfrac{5}{2}$

c. [2 points] $\int_0^1 xf(x)\,dx$

Solution: Using integration by parts we have: $\int_0^1 xf(x)\,dx = \big(xF(x)\big)\Big|_0^1 - \int_0^1 F(x)\,dx = F(1) - 0 - 3 = 1 - 3 = -2$

d. [2 points] $\int_0^1 f(2x+1)\,dx$

Solution: Using the u-substitution $u = 2x+1$ we have: $\int_0^1 f(2x+1)\,dx = \dfrac{1}{2}\int_1^3 f(u)\,du = \dfrac{1}{2}(F(3) - F(1)) = \dfrac{-4-1}{2} = -\dfrac{5}{2}$

2. [12 points] In this problem **you must give full evidence supporting your answer, showing all your work and indicating any theorems about series you use.**

a. [7 points] Show that the following series **converges**. Does it converge conditionally or absolutely? Justify.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

Solution: We notice that

$$\left|\frac{(-1)^n}{n! + 2^n}\right| = \frac{1}{n! + 2^n} \leq \frac{1}{2^n} \quad \text{for all } n \geq 1 \quad (*)$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges because it is a geometric series with ratio $1/2$ and $|1/2| < 1$. Thus, using the inequality $(*)$, the series

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n! + 2^n}\right|$$

converges by the comparison test. Since this series converges, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n! + 2^n}$$

converges absolutely. (This also shows that it converges).

b. [5 points] Determine whether the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Solution: The function $f(x) = \frac{1}{x \ln x}$ is decreasing and has $\lim\limits_{x \to \infty} f(x) = 0$. Moreover,

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^{b} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\, du = \lim_{b \to \infty} (\ln(\ln b) - \ln(\ln 2)) = \infty$$

where in the second equality we used the substitution $u = \ln x$. So the integral $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ diverges. Therefore, the integral test applies and tells us that the series in question diverges as well.

3. [12 points] In this problem we study the integral $I = \int_1^{1.5} \ln x \, dx$.

a. [2 points] Write a left Riemann sum with 5 subdivisions that approximates I, showing all the terms in your sum. **Circle** your sum and leave all the terms in **exact** form.

Solution:

$$I \approx LEFT(5) = 0.1\ln 1 + 0.1\ln 1.1 + 0.1\ln 1.2 + 0.1\ln 1.3 + 0.1\ln 1.4$$

b. [2 points] Use the midpoint rule with 5 subdivisions to approximate I, showing all the terms in your sum. **Circle** your sum and leave all the terms in **exact** form.

Solution:

$$I \approx MID(5) = 0.1\ln 1.05 + 0.1\ln 1.15 + 0.1\ln 1.25 + 0.1\ln 1.35 + 0.1\ln 1.45$$

c. [4 points] (i) Use the u-substitution $u = x - 1$ to find an integral J, which is equal to I. **Circle** your answer.

Solution: $I = \int_0^{0.5} \ln(u+1)\, du = J$

(ii) Give $P_3(u)$, the 3rd degree Taylor polynomial around $u = 0$ for the integrand of the integral J. **Circle** your answer.

Solution: $P_3(u) = u - \frac{u^2}{2} + \frac{u^3}{3}$

(iii) Substitute $P_3(u)$ for the integrand of J and compute the resulting integral by hand. **Circle** your answer.

Solution:

$$J = \int_0^{0.5} \ln(u+1)\, du \approx \int_0^{0.5} u - \frac{u^2}{2} + \frac{u^3}{3}\, du = \left(\frac{u^2}{2} - \frac{u^3}{6} + \frac{u^4}{12}\right)\Big|_0^{0.5} = \frac{1}{8} - \frac{1}{48} + \frac{1}{192} \approx 0.109$$

3. (continued)

d. [4 points] Finally find the exact value of $I = \int_1^{1.5} \ln x \, dx$ using integration by parts. Give your answer in **exact** form and show your work. **Circle** your answer.

Solution: Using integration by parts with $u = \ln x, u' = 1/x, v = x, v' = 1$ we find

$$I = \int_1^{1.5} \ln x \, dx = (x \ln x)\Big|_1^{1.5} - \int_1^{1.5} x \frac{1}{x} \, dx = 1.5 \ln 1.5 - (1.5 - 1) = 1.5 \ln 1.5 - 0.5$$

4. [5 points]

The function $g(x)$ satisfies the differential equation $y' = ay^2 - x$. The table on the right gives some information about $g(x)$.

x	$g(x)$	$g'(x)$
1	1	2

a. [2 points] Find a.

Solution: Plugging the data in the differential equation, we get $2 = a \cdot 1^2 - 1$ which gives $a = 3$.

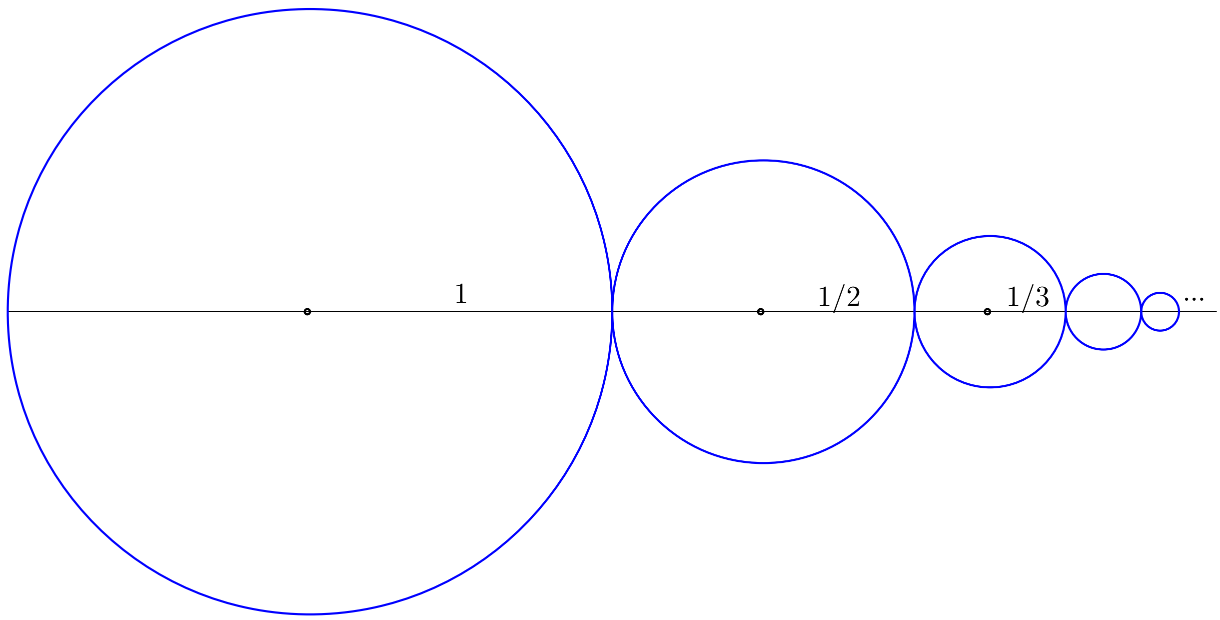
b. [3 points] Approximate $g(1.2)$ using Euler's method with $\Delta x = 0.1$.

Solution: We use the formula $\Delta y = \Delta x \cdot y'$. We calculate y' from the differential equation $y' = 3y^2 - x$.

x	y	y'
1	1	2
1.1	1.2	3.22
1.2	1.522	...

Thus, $g(1.2) \approx 1.522$.

5. [6 points] O-guk is eating pizzas! All is well now, so he got hungry. He has put them next to each other, as depicted below, so that he can devour them one after another. There are infinitely many pizzas, and they have radii $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
The following figure shows the first five pizzas.



a. [4 points] Write infinite series for the total area and the total perimeter of the pizzas. You must write your series in sigma notation.

Total area: $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{n^2}$

Total perimeter: $\displaystyle\sum_{n=1}^{\infty} \frac{2\pi}{n}$

b. [2 points] In the next two questions **circle** the correct answer.

Is the total area a finite number?

$\boxed{\textbf{YES}}$ **NO**

Is the total perimeter a finite number?

YES $\boxed{\textbf{NO}}$

6. [4 points] Determine the **exact** value of the infinite series in each of the following questions. No decimal approximations are allowed. You do not need to show your work. **Circle** your answer.

a. [2 points] $\frac{1}{5^2} - \frac{1}{5^4} + \frac{1}{5^6} - \frac{1}{5^8} + \frac{1}{5^{10}} - \frac{1}{5^{12}} + \cdots = \frac{\frac{1}{5^2}}{1 - (-\frac{1}{5^2})} = \frac{1}{26}$

b. [2 points] $\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n}}{(2n+1)!} = \frac{1}{5}\sin(5)$

7. [6 points] Consider the differential equation $y' = 1 - 2xy$.

a. [4 points] Suppose k is an arbitrary constant. Show that the function

$$y(x) = \frac{k + \int_2^x e^{t^2}\, dt}{e^{x^2}}$$

is a solution to the differential equation.

Solution: We want to show that $y'(x) = 1 - 2xy(x)$. We show this by showing that both sides are equal to the same quantity:

$$y'(x) = \frac{e^{x^2} e^{x^2} - (k + \int_2^x e^{t^2}\, dt) \cdot e^{x^2} 2x}{e^{2x^2}} = 1 - \frac{(k + \int_2^x e^{t^2}\, dt) \cdot 2x}{e^{x^2}}$$

where in the first equality we used both the 2nd Fundamental Theorem and the quotient rule.

On the other hand,

$$1 - 2xy(x) = 1 - 2x\frac{k + \int_2^x e^{t^2}\, dt}{e^{x^2}}.$$

Thus, $y(x)$ satisfies the differential equation.

b. [2 points] Give the value of k so that the graph of the solution to the differential equation passes through the point $(2, 7)$.

Solution: We have

$$y(2) = 7$$
$$\frac{k}{e^4} = 7$$
$$k = 7e^4.$$

8. [9 points] Consider the function $F(t)$ defined by the power series

$$F(t) = \sum_{n=0}^{\infty} \frac{n(t-5)^{2n}}{3^n(n+1)}.$$

a. [6 points] Find the **radius** of convergence of the power series. Show all your work.

Solution:

$$\lim_{n\to\infty} \frac{\left|\frac{(n+1)(t-5)^{2(n+1)}}{3^{n+1}(n+2)}\right|}{\left|\frac{n(t-5)^{2n}}{3^n(n+1)}\right|} = \lim_{n\to\infty} \frac{(n+1)^2 3^n (t-5)^{2n+2}}{n(n+2)3^{n+1}(t-5)^{2n}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2(t-5)^2}{n(n+2)3} = \frac{(t-5)^2}{3}$$

By the ratio test, we have

$$\frac{(t-5)^2}{3} < 1$$

$$(t-5)^2 < 3$$

$$|t-5| < \sqrt{3}$$

Thus, the radius of convergence is $R = \sqrt{3}$.

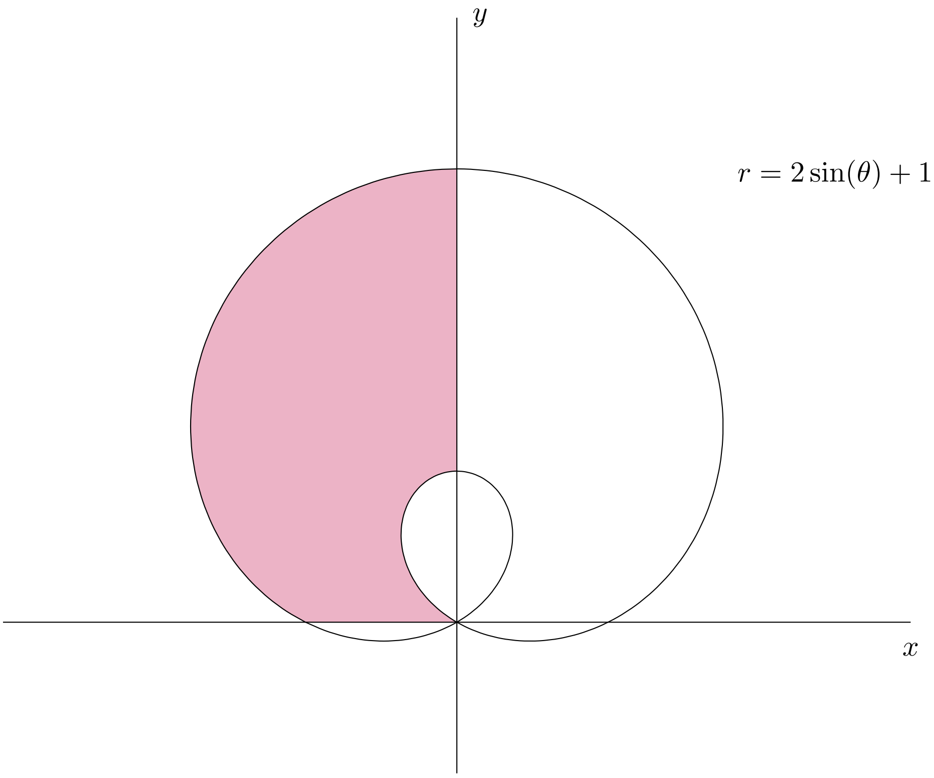
b. [3 points] Calculate $F^{(40)}(5)$. Give an **exact** answer.

Solution: By the theory of power series, we know that the coefficient of the term $(t-5)^{40}$ is given by $\frac{F^{(40)}(5)}{40!}$. In our case, this coefficient is given by $\frac{n}{3^n(n+1)}$ for $n = 20$. Therefore,

$$\frac{F^{(40)}(5)}{40!} = \frac{20}{3^{20} \cdot 21}$$

$$F^{(40)}(5) = \frac{20 \cdot 40!}{21 \cdot 3^{20}}$$

9. [6 points] O-guk is creating a can opener to open his many cans of juice. The opener is in the shape of the shaded region enclosed by the two loops of the polar curve $r = 2\sin(\theta) + 1$ and the x- and y-axes.

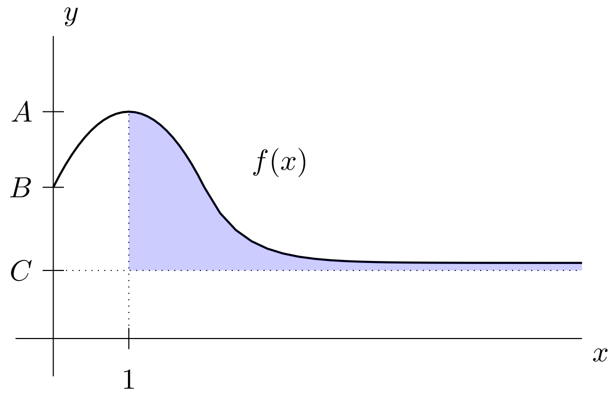


Write an expression involving integrals that gives the total area of the shaded region.

Solution:

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (2\sin\theta + 1)^2\, d\theta - \frac{1}{2}\int_{\frac{3\pi}{2}}^{\frac{11\pi}{6}} (2\sin\theta + 1)^2\, d\theta$$

10. [14 points] A function f has domain $[0, \infty)$, and its graph is given below. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



a. [5 points] Determine the convergence of the improper integral below. **You must give full evidence supporting your answer, showing all your work and indicating any theorems about integrals you use**.

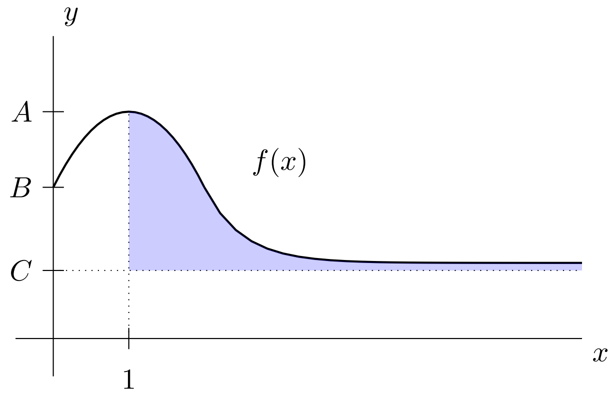
$$\int_0^1 \frac{f(x)}{x}\, dx$$

Solution: We have that for $0 < x \leq 1$

$$\frac{f(x)}{x} \geq \frac{B}{x}$$

The improper integral $\displaystyle\int_0^1 \frac{B}{x}\, dx = B\int_0^1 \frac{1}{x}\, dx$ diverges by the p-test with $p = 1$. Thus, the integral $\displaystyle\int_0^1 \frac{f(x)}{x}\, dx$ diverges by the comparison test.

10. (continued) For your convenience, the graph of f is given again. The numbers A, B, C are positive constants. The shaded region has **finite area**, but it extends infinitely in the positive x-direction. The line $y = C$ is a horizontal asymptote of $f(x)$ and $f(x) > C$ for all $x \geq 0$. The point $(1, A)$ is a local maximum of f.



b. [3 points] **Circle** the correct answer. The value of the integral $\displaystyle\int_1^\infty f(x)f'(x)\,dx$

is $C - A$ $\quad$ $\boxed{\text{is } \frac{C^2-A^2}{2}}$ $\quad$ is $B - A$ $\quad$ cannot be determined $\quad$ diverges

c. [3 points] **Circle** the correct answer. The value of the integral $\displaystyle\int_1^\infty f'(x)\,dx$

$\boxed{\text{is } C - A}$ $\quad$ is $\frac{C^2-A^2}{2}$ $\quad$ is C $\quad$ cannot be determined $\quad$ diverges

d. [3 points] Determine, with justification, whether the following series converges or diverges.

$$\sum_{n=1}^{\infty}(f(n) - C)$$

Solution: We notice that the function $f(x) - C$ is decreasing, positive with $\lim_{x\to\infty}(f(x) - C) = 0$. By the integral test, the series

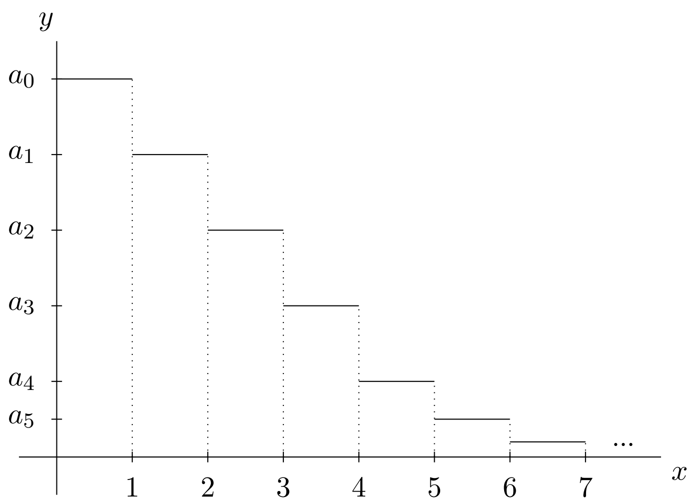
$$\sum_{n=1}^{\infty}(f(n) - C)$$

converges if and only if the improper integral

$$\int_1^\infty (f(x) - C)$$

converges. But this integral gives exactly the shaded area, which we know that it is finite. So this integral converges and therefore the series converges as well.

11. [5 points] The Hanoi tower is made by rotating the region depicted below around the y-axis. The region is made up of infinitely many adjacent rectangles. The nth rectangle has width 1 and height $a_n = \dfrac{1}{n!(2n+1)}$ where $n = 0, 1, 2, 3, \ldots.$ The rectangle touching the y-axis corresponds to $n = 0$. Note that the y-axis is not to scale.



Compute the volume of the Hanoi Tower. Give an **exact** answer.

Solution: To compute the volume of the Hanoi Tower, we focus on each rectangle separately. The volume of the object made by the revolution of the nth rectangle is given by

$$[\pi(n+1)^2 - \pi n^2] \cdot a_n = \pi(2n+1)\frac{1}{n!(2n+1)} = \frac{\pi}{n!}$$

The total volume is given by adding the volume of all those objects for $n = 0, 1, 2, 3, \ldots.$

$$\sum_{n=0}^{\infty} \frac{\pi}{n!} = \pi \cdot \sum_{n=0}^{\infty} \frac{1}{n!} = \pi e$$

12. [8 points] Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges when $x = 0$ and diverges when $x = 9$. In this problem, you do not need to show your work.

a. [4 points] Which of the following could be the interval of convergence? Circle **all** that apply.

$\boxed{[0,8]}$ $\qquad [0,7] \qquad$ $\boxed{(-1,9)}$ $\qquad (-2,10) \qquad (0,8]$

b. [2 points] The limit of the sequence a_n is 0.

$\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

c. [2 points] The series $\sum_{n=0}^{\infty} (-5)^n a_n$ converges.

ALWAYS $\boxed{\textbf{SOMETIMES}}$ **NEVER**

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$