On my honor, as a student,
I have neither given nor received
unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — Final Exam — April 24, 2017

Your Initials Only: ______ Your U-M ID # (not uniqname): ____________________

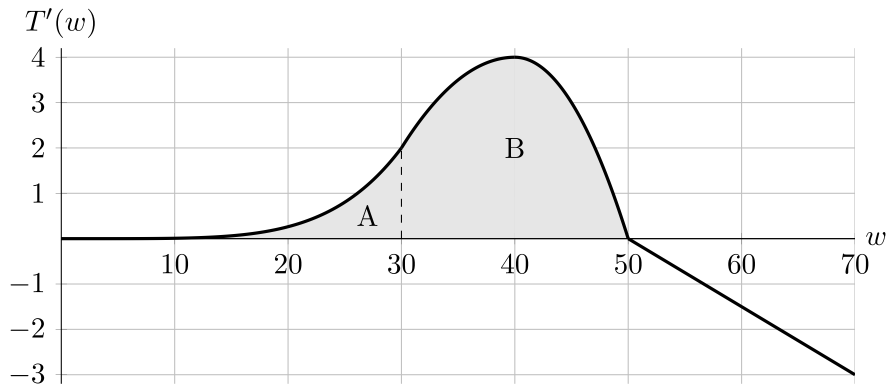
Instructor Name: ______________________________ Section #: __________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 13 pages including this cover. There are 12 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	7	
3	9	
4	7	
5	10	
6	7	

Problem	Points	Score
7	6	
8	8	
9	8	
10	8	
11	12	
12	12	
Total	100	

1. [6 points] Suppose that the yield of the tomato plants in a particular Michigan garden is a function of the amount of water that the plants receive (from rainfall and irrigation).
Let $T(w)$ be the seasonal yield (in pounds) of the tomato plants in a season when the plants receive w gallons of water every week. A portion of the graph of $T'(w)$ (the derivative of $T(w)$) is shown below.
Note that $T'(w)$ is linear for $50 \le w \le 70$. Let A be the area of the region between the w–axis and the graph of $T'(w)$ for $0 \le w \le 30$, and let B be the area of the region between the w–axis and the graph of $T'(w)$ for $30 \le w \le 50$,



a. [2 points] If the tomato plants yield 150 pounds of tomatoes when the plants receive 70 gallons of water every week, how many pounds of tomatoes would the plants yield in a season when they receive 30 gallons of water each week? (Your answer may involve the constants A and B.)

Answer: ______________________________

b. [2 points] In order to maximize the yield of the tomato plants, how many gallons of water should the plants receive each week? (Round to the nearest 5 gallons.)

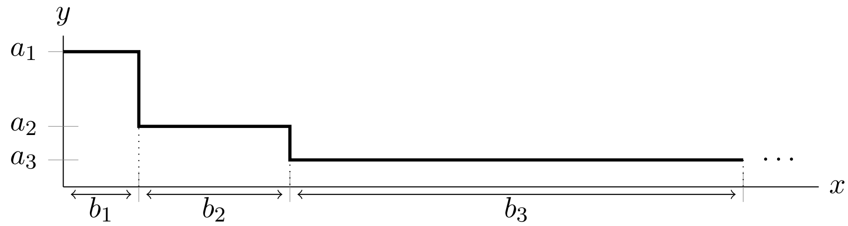
Answer: ______________________________

c. [2 points] Consider the integral $\displaystyle\int_{10}^{30} T'(w)\,dw$.
Rank the following four estimates of the value of this integral in order from least to greatest by writing them in the correct order on the answer blanks below:

LEFT(10) RIGHT(10) TRAP(10) MID(10)

__________ < __________ < __________ < __________

2. [7 points] The region depicted below consists of infinitely many adjacent rectangles. (Only the first three rectangles are actually shown, and they are not necessarily drawn to scale.) For $n = 1, 2, 3, \ldots$, the nth rectangle has height $a_n = \dfrac{1}{5^{n/2}}$ and width $b_n = n!$.



a. [5 points] Write an infinite series that gives the total volume of the solid formed by rotating the entire region (all of the rectangles) around the x-axis.

b. [2 points] Does the infinite series that gives the total volume of the solid formed by rotating the entire region (all of the rectangles) around the x-axis converge or diverge?

CIRCLE ONE: **Converges** **Diverges**

State the name of the test you would use to justify your answer. If you would use the comparison test or limit comparison also give a valid comparison series. You do not need to actually write out a full justification. (If you do not know the name of the test you would use, state the test itself.)

3. [9 points] In this problem you must give full evidence supporting your answer, showing all your work and indicating any theorems or tests about series you use. (Remark: You **cannot** use any results about convergence from the team homework without justification.)

a. [4 points] Determine whether the series below converges or diverges, and circle your answer clearly. Justify your answer as described above.

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$$

Converges **Diverges**

b. [5 points] Determine if the following infinite series converges absolutely, converges conditionally, or diverges, and circle your answer clearly. Justify your answer as described above.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}$$

Converges Absolutely **Converges Conditionally** **Diverges**

4. [7 points] The position of a particle in the plane is given by a pair of parametric equations $x = x(t)$ and $y = y(t)$ where x and y are measured in meters and t is measured in seconds. The functions $x(t)$ and $y(t)$ satisfy the differential equations

$$\frac{dx}{dt} = p(x) \qquad \text{and} \qquad \frac{dy}{dt} = h(t)$$

for functions $p(x)$ and $h(t)$. Some values of the functions p and h are provided in the tables below.

x	0	1	2	3	4	5
$p(x)$	1	4	6	-2	0	3

t	0	0.5	1	1.5	2
$h(t)$	2	-4	1	0	3

a. [4 points] Suppose that $x(0) = 3$ and $y(0) = 2$. Use Euler's method with $\Delta t = 0.5$ to approximate the values of $x(1)$ and $y(1)$. Show your calculation for each step of Euler's method.

Answer: $x(1) \approx$ ____________________

$y(1) \approx$ ____________________

b. [3 points] Suppose that $x(2) = 1$ and $y(2) = 5$. How fast is the particle moving when $t = 2$?

5. [10 points] For each of the problems below, fill in the letter (A-I) corresponding to the answer from the list of answer choices below which correctly finishes the sentence. If none of those options is correct, choose "NONE OF THESE". Note that answer choices may be used more than once.

Answer Choices:

A. equals 0	D. equals 3	G. equals 6
B. equals 1	E. equals 4	H. equals 7
		I. diverges
C. equals 2	F. equals 5	J. NONE OF THESE

a. [2 points] The integral $\int_0^\infty 5xe^{-x}\,dx$

Answer: ______________

b. [2 points] The series $(e-1) - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(e-1)^n}{n} + \cdots$

Answer: ______________

c. [2 points] The arclength of the polar curve $r = \frac{\sin(\theta)}{\pi}$

Answer: ______________

d. [2 points] The TRAP(2) estimate for the integral $\int_0^1 8x^2\,dx$

Answer: ______________

e. [2 points] The series $\sum_{n=3}^{\infty}\left(\frac{12}{n} - \frac{12}{n+2}\right)$

Answer: ______________

6. [7 points] Consider the function F defined by $F(x) = \displaystyle\int_{-\infty}^{x} e^{-t^2}\, dt$.

For each value of x, the right hand side is an improper integral that converges. The function $F(x)$ is an antiderivative of e^{-x^2}. (You do not need to verify this.)

a. [4 points] It can be shown that $F(0) = \dfrac{\sqrt{\pi}}{2}$. Using this fact, write the first four nonzero terms of the Taylor series for the function $F(x)$ centered at $x = 0$.

b. [3 points] Use your answer from part **a.** to approximate the integral $\displaystyle\int_{0}^{1} e^{-t^2}\, dt$.

7. [6 points]

a. [3 points] Find the interval of convergence of the power series $\displaystyle\sum_{k=3}^{\infty} \frac{x^k}{(k^2+1)3^k}$.

You do not need to show your work.

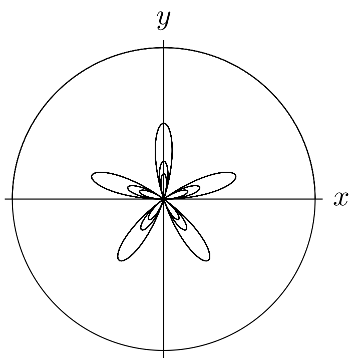
Answer: Interval of Convergence = ________________________________

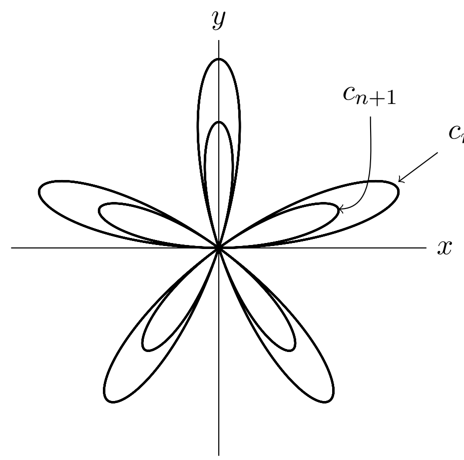
b. [3 points] Consider the power series $\displaystyle\sum_{j=0}^{\infty} C_j (x-2)^j$.

This power series converges when $x = -1$ and diverges when $x = 7$.
Which, if any, of the following intervals could be exactly equal to the interval of convergence for this power series? Circle all the intervals below that could be exactly equal to the interval of convergence or circle "NONE OF THESE".

$[-2, 6]$ $[-2, 7)$ $[-3, 7)$ $(-3, 7]$ $[-1, 4]$ $[-4, 7)$ NONE OF THESE

8. [8 points] A very unique sand dollar has an interesting pattern on it. The outline of the sand dollar is given by the polar equation $r = 2$. On the face of the sand dollar are many other polar curves c_n. For $n = 1, 2, 3, \ldots$, the curve c_n is given by the polar equation $r = \frac{1}{n}\sin(5\theta)$. Below is a picture of the sand dollar (left) and a zoomed in view of two of the polar curves (right).





a. [3 points] Let a_0 be the area that is inside the sand dollar but outside c_1. Write a formula involving one or more integrals for a_0.

b. [3 points] For $n \geq 1$, let a_n be the area inside c_n but outside c_{n+1}. Write a formula involving one or more integrals for a_n when $n \geq 1$.

c. [2 points] Does the infinite series $\sum_{n=0}^{\infty} a_n$ converge or diverge? If it converges, what is its value? No justification is necessary.

9. [8 points] This problem concerns a rocket that has been launched and is ascending.
You may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

Because it is burning fuel, the rocket's mass is decreasing. Let $m(h)$ be the mass (in kg) of the rocket during its ascent when it is at a height of h meters above the ground.

a. [2 points] Suppose Δh is small. Write an expression (not involving integrals) in terms of m and h that approximates the work (in joules) required for the rocket to ascend from a height of h meters above the ground to a height of $h + \Delta h$ meters above the ground.

b. [2 points] Write, but do **not** evaluate, an integral that gives the total work (in joules) required for the rocket to ascend from a height of 100 meters above the ground to a height of 2500 meters above the ground.

Let $v(h)$ be the rocket's velocity (in m/s) when it is at a height of h meters above the ground.

c. [2 points] Suppose Δh is small. Write an expression (not involving integrals) in terms of v and h that approximates the time (in seconds) it takes for the rocket to ascend from a height of h meters above the ground to a height of $h + \Delta h$ meters above the ground.

d. [2 points] Write, but do **not** evaluate, an integral that gives the total time (in seconds) it takes for the rocket to ascend from a height of 100 meters above the ground to a height of 2500 meters above the ground.

10. [8 points] The Taylor series centered at $x = 0$ for a function $F(x)$ converges to $F(x)$ for all x and is given below.

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}$$

a. [3 points] What is the value of $F^{(101)}(0)$?
Make sure your answer is exact. You do not need to simplify.

Answer: $F^{(101)}(0) =$ ____________________________________

b. [3 points] Find $P_9(x)$, the 9th degree Taylor polynomial that approximates $F(x)$ near $x = 0$.

c. [2 points] Use your Taylor polynomial from part **b.** to compute

$$\lim_{x \to 0} \frac{F(x) - x}{2x^5}$$

11. [12 points] Quinn is a patient taking a new experimental medicine.

a. [4 points] Quinn knows that the amount of the medicine in her body decays at a rate proportional to the current amount of the medicine in her body with constant of proportionality $k > 0$. Let $Q = Q(t)$ be the quantity, in mg, of this medicine that is in Quinn's body t days after she begins taking it. Assuming the medicine enters her body at a continuous rate of 200mg per day, write a differential equation that models $Q(t)$, and give an appropriate initial condition.

Answer: Differential Equation: ______________________

Initial Condition: ______________________

For parts **b.-d.** below, suppose that the medicine has a half-life of 12 hours in her body and that, rather than entering her body continuously throughout the day, Quinn takes one 200mg pill each morning at 8am.

Let Q_n be the quantity, in mg, of this medicine that is in Quinn's body immediately after she takes the nth pill. For example, Q_1 is the amount of medicine in her body immediately after she takes her first dose.

b. [2 points] Find the values of Q_1, Q_2 and Q_3.

Answers: $Q_1 =$ __________ $Q_2 =$ __________ $Q_3 =$ __________

c. [4 points] Write a closed form expression for Q_n. (Your answer should not include sigma notation or ellipses $(\cdots)$.)

Answer: $Q_n =$ ______________________

d. [2 points] What is $\lim_{n\to\infty} Q_n$? Interpret your answer in the context of the problem.

12. [12 points] For each of the statements and questions below, circle all of the available choices that correctly complete the statement or answer the question.
Circle "NONE OF THESE" if none of the available choices are correct.
No justification is required. No credit will be awarded for unclear markings.

a. [3 points] Let $a_n = \int_{\frac{1}{n}}^{1} \ln(x)\,dx$. Then the sequence a_n

i. diverges

ii. converges to -1

iii. is bounded

iv. is increasing

v. NONE OF THESE

b. [3 points] Suppose $K(x) = \int_{e}^{\sin(x)} e^t \sin(t)\,dt$. Then $K'(x)$ is equal to

i. $e^{\sin(x)} \sin(\sin(x)) \cos(x)$

ii. $e^{\sin(x)} \sin(\sin(x)) \cos(x) - e^e \sin(e) e$

iii. $e^{\sin(x)} \sin(x)$

iv. $e^{\sin(x)} \sin(\sin(x))$

v. NONE OF THESE

c. [3 points] The radius of convergence of the Taylor series centered at $w = 0$ for the function $f(w) = (1 + 3w^2)^{1/3}$ is

i. $\sqrt{3}$ ii. $\dfrac{1}{3}$ iii. $\dfrac{1}{\sqrt{3}}$ iv. $\left(\dfrac{1}{3}\right)^{1/3}$ v. NONE OF THESE

d. [3 points] If k is a constant with $k > 1$, for which of the following series does the series definitely converge with sum equal to k?

i. $1 + \ln(k) + \dfrac{(\ln(k))^2)}{2!} + \dfrac{(\ln(k))^3}{3!} + \dfrac{(\ln(k))^4}{4!} + \cdots$

ii. $2k - \dfrac{2k(\pi/3)^2}{2!} + \dfrac{2k(\pi/3)^4}{4!} - \dfrac{2k(\pi/3)^6}{6!} + \cdots$

iii. $k - 1 + \dfrac{k-1}{k} + \dfrac{k-1}{k^2} + \dfrac{k-1}{k^3} + \cdots$

iv. $k(\pi/2) - \dfrac{k(\pi/2)^3}{3!} + \dfrac{k(\pi/2)^5}{5!} - \cdots$

v. NONE OF THESE

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$