# Math 116 — First Midterm — February 6, 2017

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

| Problem | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 13 | |
| 3 | 12 | |
| 4 | 12 | |
| 5 | 12 | |

| Problem | Points | Score |
|---|---|---|
| 6 | 10 | |
| 7 | 10 | |
| 8 | 8 | |
| 9 | 11 | |
| Total | 100 | |

**1.** [12 points] Suppose that $f$ is a twice differentiable function with continuous second derivative. (That is, both $f$ and $f'$ are differentiable, and $f''$ is continuous.) The following table gives some values of $f$ and $f'$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $e^3$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 7 | 5 | $-1$ | 0 | 11 | $-3$ | 2 | 9 |
| $f'(x)$ | 3 | $-4$ | $-2$ | 4 | $-5$ | 0 | $-1$ | 2 |

In parts (a) through (c) below, calculate the exact numerical value of the integral. Write "NOT ENOUGH INFO" if there is not enough information to find the exact value. Be sure to show your work clearly. No partial credit will be given for estimates.

**a.** [4 points] $\displaystyle\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx$

*Solution:* The substitution $w = \ln(x)$ gives $dw = \frac{dx}{x}$ and

$$\int_1^{e^3} \frac{f'(\ln x)}{x}\,dx = \int_0^3 f'(w)\,dw = f(3) - f(0) = 0 - 7 = -7.$$

**b.** [4 points] $\displaystyle\int_0^4 x f''(x)\,dx$

*Solution:* Integration by parts with $u = x$ and $dv = f''(x)\,dx$ gives

$$\begin{aligned}\int_0^4 x f''(x)\,dx &= x f'(x)\Big|_0^4 - \int_0^4 f'(x)\,dx \\ &= \big(4\cdot f'(4) - 0\cdot f'(0)\big) - \big(f(4) - f(0)\big) \\ &= -20 - (11 - 7) = -24.\end{aligned}$$

**c.** [4 points] $\displaystyle\int_2^6 f'(x)\,[f(x)]^2\,dx$

*Solution:*
One Approach: substitution with $w = f(x)$ so $dw = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = \int_{f(2)}^{f(6)} w^2\,dw = \frac{w^3}{3}\Big|_{-1}^{2} = \frac{8}{3} - \frac{-1}{3} = 3.$$

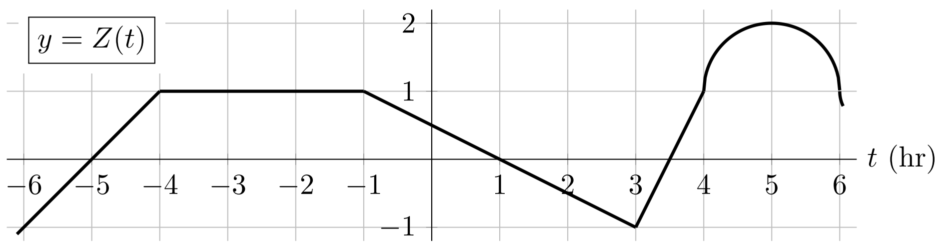
Another Approach: integration by parts with $u = f(x)^2$ and $dv = f'(x)\,dx$

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 - 2\int_2^6 f'(x)[f(x)]^2\,dx.$$

Moving the last term to the left hand side and dividing both sides of the resulting equation by 3 gives

$$\int_2^6 f'(x)\,[f(x)]^2\,dx = [f(x)]^3\big|_2^6 = \frac{8-(-1)}{3} = 3.$$

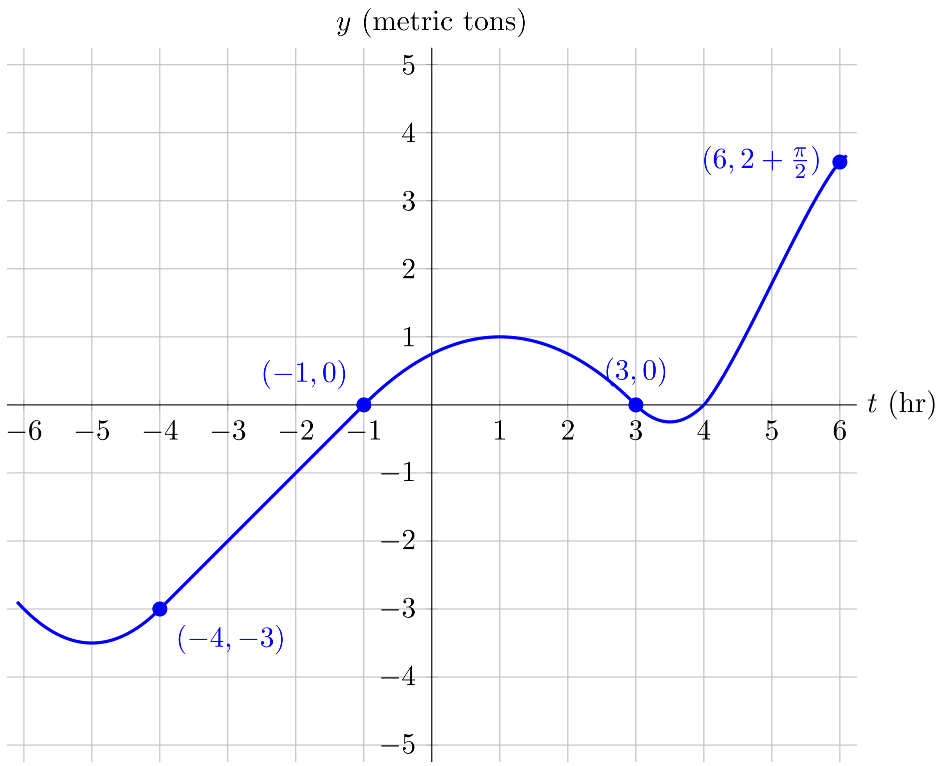
**2**. [13 points] Suppose $Z(t)$ is the rate of change, in metric tons per hour, of the biomass (i.e. total mass) of zooplankton in Loch Ness $t$ hours after 8am on January 25, 2017.
Below is a portion of the graph of $Z(t)$. Note that this graph is linear on the intervals $[-6,-4]$, $[-4,-1]$, $[-1,3]$, and $[3,4]$. Also note that the portion of the graph for $4 \le t \le 6$ is the upper half of a circle centered at the point $(5,1)$.




Let $B(t)$ be the biomass, in metric tons, of zooplankton in Loch Ness $t$ hours after 8am on January 25, 2017.

**a**. [10 points] Carefully sketch a graph of $y = B(t) - B(3)$ for $-6 \le t \le 6$ using the axes provided below. If there are features of this function that are difficult for you to draw, indicate these on your graph. Be sure that local extrema and concavity are clear.
Label the coordinates of the points on your graph at $t = -4, -1, 3, 6$.

*Solution:* Note that $B(t) - B(3) = \int_3^t Z(x)\,dx$ is the antiderivative of $Z(t)$ whose value is 0 at $t = 3$. Although it is difficult to tell here, the graph below is concave up for $4 < t < 5$ and concave down for $5 < t < 6$.

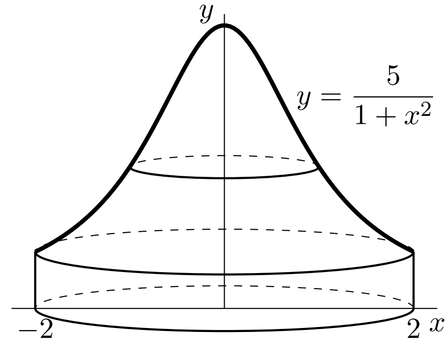



**b**. [3 points] Define $A(h)$ to be the average biomass (in metric tons) of zooplankton in Loch Ness during the first $h$ hours after 8am on January 25, 2017. Write an expression for $A(h)$. (Your expression may involve integrals, the function $Z$, and/or the function $B$.)

*Solution:* $A(h)$ is the average value of the function $B(t)$ over the interval $0 \le t \le h$, so

$$A(h) = \frac{1}{h}\int_0^h B(t)\,dt.$$

**3**. [12 points] A new top secret weather balloon has the ability to make it rain orange soda. The base of the balloon is a solid cylinder with a radius of 2 meters and a height of 1 meter. Above that is a solid obtained by taking the portion of the function $y = \dfrac{5}{1+x^2}$ for $0 \le x \le 2$ and rotating it around the $y$-axis (where $x$ and $y$ are measured in meters). The balloon is made of a light metal which has a constant density $\delta$ kg/m$^3$. The balloon is pictured on the right. (The picture is not to scale.)




**a**. [4 points] Write down an expression in terms of $y$ (but not $x$) that approximates the volume, in cubic meters, of a horizontal slice of the weather balloon of thickness $\Delta y$ at a height $y$ meters above the ground where $1 < y < 5$.

*Solution:* Solving the equation $y = \frac{5}{1+x^2}$ for $x$ gives the radius of a circular cross section at height $y$. The volume of the slice is then approximately $\pi\left(\frac{5}{y} - 1\right) \cdot \Delta y$ cubic meters

**b**. [4 points] Write down an expression involving one or more integrals which gives the total mass of the weather balloon in kilograms. Do **not** evaluate any integrals in this expression.

*Solution:* We integrate the solution from part (a) from 1 to 5 replacing $\Delta y$ with $dy$ and then add $4\pi$ (the volume of the cylindrical base) to get the total volume of the balloon. Finally we multiply this entire expression by the constant density $\delta$ to obtain a total mass of

$$4\pi\delta + \int_1^5 \delta\pi\left(\frac{5}{y} - 1\right)\,dy \quad \text{kilograms.}$$

**c**. [4 points] Write down an expression involving one or more integrals which gives the $y$–coordinate of the center of mass of the weather balloon. Do **not** evaluate any integrals in this expression.

*Solution:* We rewrite the first term in (b) as $\displaystyle\int_0^1 4\pi\,dy$ and include a factor of $y$ in each integral and then divide by the total mass to find that the $y$–coordinate of the center of mass is

$$\frac{\displaystyle\int_0^1 y\cdot 4\pi\delta\,dy + \int_1^5 y\cdot\delta\pi\left(\frac{5}{y} - 1\right)\,dy}{\displaystyle 4\pi\delta + \int_1^5 \delta\pi\left(\frac{5}{y} - 1\right)\,dy}.$$

**4**. [12 points] For each of the questions below, circle all of the available correct answers. Circle "NONE OF THESE" if none of the available choices are correct.

**a**. [3 points] Which of the following are antiderivatives of the function $2\sin(x)\cos(x)$?

i. $\frac{1}{2}\cos^2(x) + \frac{1}{2}\sin^2(x)$

ii. $\boxed{\sin^2(3) - \cos^2(x)}$

iii. $\displaystyle\int_0^{\pi} 2\sin(x)\cos(x)\,dx$

iv. $\boxed{\sin^2(x)}$

v. NONE OF THESE

**b**. [3 points] Which of the following integrals give the arc length of the curve $y = e^{2x}$ from $x = 0$ to $x = 2$?

i. $\displaystyle\int_0^2 \sqrt{1+4e^{2x}}\,dx$

ii. $\displaystyle\int_0^2 \sqrt{1+e^{4x}}\,dx$

iii. $\displaystyle\frac{1}{2}\int_0^1 \sqrt{1+4e^{2s}}\,ds$

iv. $\boxed{\displaystyle\int_0^2 \sqrt{1+4e^{4u}}\,du}$

v. NONE OF THESE

**c**. [3 points] Which of the following are antiderivatives of the function $\dfrac{1}{\ln x}$?

i. $\ln(\ln(x)) + 4$

ii. $\displaystyle\int_2^e \frac{1}{\ln t}\,dt$

iii. $\boxed{\displaystyle\int_1^{\ln x} \frac{e^t}{t}\,dt}$

iv. $\boxed{\displaystyle\int_2^x \frac{1}{\ln t}\,dt}$

v. NONE OF THESE

**d**. [3 points] An object with variable mass is lifted up 30 meters at a constant rate. This process takes 10 seconds. Suppose that $m(t)$ is the mass of the object, in kilograms, $t$ seconds after the lifting begins. Let $g$ be the acceleration due to gravity in m/s$^2$. (So $g \approx 9.8$.) Which of the following expressions give the work, in joules, required to raise the object?

i. $\boxed{\displaystyle 3\int_0^{10} g \cdot m(t)\,dt}$

ii. $\boxed{\displaystyle\int_0^{30} g \cdot m\left(\frac{x}{3}\right)\,dx}$

iii. $\displaystyle\frac{1}{3}\int_0^{30} g \cdot m(x)\,dx$

iv. $\displaystyle\int_0^{10} g \cdot 3t \cdot m(t)\,dt$

v. NONE OF THESE

**5**. [12 points] Several high-ranking Illuminati officials are relaxing while counting their money. There is so much money to count that the process takes many hours. The rate (in millions of dollars per hour) at which they count the money is given by the function $M(t)$, where $t$ is the number of hours since they began counting. Several values of this function are given in the table below.

| $t$ (hours) | 0 | 2 | 4 | 6 | 10 | 14 |
|---|---|---|---|---|---|---|
| $M(t)$ (million \$/hour) | 5 | 6 | 11 | 12 | 10 | 3 |

Note: The function $M(t)$ is continuous. Between each of the values of $t$ given in the table, the function $M(t)$ is always increasing or always decreasing.

**a**. [2 points] Write, but do **not** evaluate, a definite integral that gives the total amount of money, in millions of dollars, the officials counted from the time they started until the time when they were counting the money the fastest.

*Solution:* $\displaystyle\int_0^6 M(t)\,dt$

**b**. [3 points] Write out the terms of a left Riemann sum with 3 equal subdivisions to estimate the integral from (a). Does this sum give an overestimate or an underestimate of the integral?

*Solution:* This left Riemann sum is $2 \cdot 5 + 2 \cdot 6 + 2 \cdot 11 = 44$.
Since $M(t)$ is increasing on the interavel $[0, 6]$, this is an underestimate of the integral from (a).

**c**. [4 points] Based on the data provided, write a sum that gives the best possible **overestimate** for the total amount of money, in millions of dollars, counted during the first 14 hours of counting.

*Solution:* The total amount of money, in millions of dollars, counted during the first 14 hours of counting is equal to $\displaystyle\int_0^{14} M(t)\,dt$.
The best possible overestimate of this integral is given by the Riemann sum that uses right endpoints on the intervals over which $f$ is increasing and left endpoints on the intervals over which $f$ is decreasing. This gives the estimate $2 \cdot (6 + 11 + 12) + 4 \cdot (12 + 10)$.

**d**. [3 points] What is the difference, in millions of dollars, between the best possible overestimate and the best possible underestimate for the total amount of money counted during the first 14 hours of counting?

*Solution:* We calculate the difference between the left and right hand sums on the intervals over which $f$ is increasing and decreasing respectively and add these together, giving a difference of $2\,|12 - 5| + 4\,|3 - 12| = 50$.

**6**. [10 points] The Rodin Coil is a fantastic device that (supposedly) creates unlimited free energy. The rate at which it creates this energy is a function of the volume of the coil.

**a**. [5 points] Suppose a prototype of the Rodin Coil is the solid whose base is the circle $x^2+y^2 = 2$ (where $x$ and $y$ are measured in meters), and whose cross sections perpendicular to the $x$–axis are squares. Write, but do **not** compute, an expression involving one or more integrals which gives the volume, in cubic meters, of this prototype.

*Solution:* We find that the sidelength of the square at $x$–coordinate $x$ is $2 \cdot \sqrt{2 - x^2}$. So the volume of a slice of thickness $\Delta x$ at that point is approximately $\left(2 \cdot \sqrt{2 - x^2}\right)^2 \cdot \Delta x$. Integrating from the left end of the circle to the right end we find a total volume (in cubic meters) of

$$\int_{-\sqrt{2}}^{\sqrt{2}} \left(2\sqrt{2 - x^2}\right)^2 \, dx.$$

**b**. [5 points] One of Rodin's students was able to come up with an even more efficient free energy machine. Suppose the student's prototype was made by taking the same circle $x^2 + y^2 = 2$ and rotating it around the vertical line $x = 3$. Write, but do **not** compute, an expression involving one or more integrals which gives the volume, in cubic meters, of this prototype.

*Solution:*

One Solution: Using cylindrical shells perpendicular to the $x$–axis, we find that the volume is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} 2\pi(3 - x) \left(2\sqrt{2 - x^2}\right) \, dx.$$

Another Solution: Using slices of the solid perpendicular to the $y$–axis ("washers"), we find that the volume is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} \pi \left[\left(3 + \sqrt{2 - y^2}\right)^2 - \left(3 - \sqrt{2 - y^2}\right)^2\right] \, dy.$$

**7.** [10 points] Consider the function $F$ defined for all $x$ by the formula

$$F(x) = \int_7^{x^2} e^{-t^2}\, dt.$$

**a.** [1 point] Find a number $a \geq 0$ so that $F(a) = 0$.

*Solution:* $a = \sqrt{7}$.

**b.** [4 points]

(i) Calculate $F'(x)$. Your answer should not contain any integrals.

*Solution:* Applying the Second Fundamental Theorem of Calculus and the Chain Rule, we find

$$F'(x) = e^{-(x^2)^2} \cdot 2x = 2xe^{-x^4}.$$

(ii) Is $F(x)$ increasing on the entire interval $[1, 8]$? Why or why not?

*Solution:* $F'(x) > 0$ if $x > 1$ (in fact, if $x > 0$). Thus $F(x)$ is increasing on this interval. Alternatively, $F(x)$ is the integral of a positive function, and the interval expands as $x$ increases.

**c.** [3 points] Write out each term of a MID(3) estimate of $F(5)$.
(You do **not** need to find or approximate the numerical value of your answer.)

*Solution:* $F(5) = \displaystyle\int_7^{25} e^{-x^2}\, dx$, and therefore

$$MID(3) = 6 \cdot (e^{-10^2} + e^{-16^2} + e^{-22^2}) = 6(e^{-100} + e^{-256} + e^{-484}).$$

**d.** [2 points] Is your answer to part (c) an overestimate or underestimate of $F(5)$?
Briefly explain your reasoning.

*Solution:* Since $e^{-x^2}$ is concave up on the interval $[7, 25]$, MID(3) is an underestimate of $F(5)$.

**8.** [8 points] For each of the following, determine whether the statement is always, sometimes, or never true. Indicate your answer by circling the <u>one</u> word that correctly fills the answer blank. No justification is necessary. No credit will be awarded for unclear markings.

**a.** [2 points] If the average value of the force $F(x)$ on the interval $3 \leq x \leq 8$ is 12 N then the work done in moving a particle from $x = 3$ to $x = 8$ is __________ 60 J.

$\boxed{\text{Always}}$ Sometimes Never

**b.** [2 points] Let $k(x)$ be an even function and let $K(x)$ be an antiderivative of $k(x)$. Then $K(x)$ is __________ an odd function.

Always $\boxed{\text{Sometimes}}$ Never

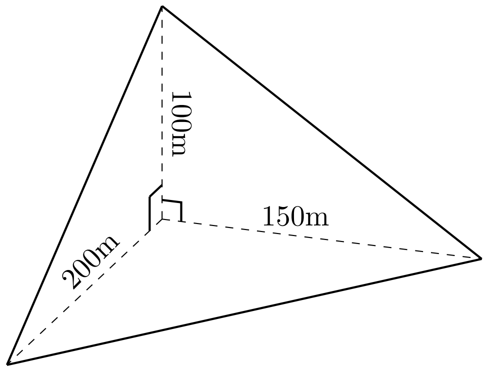
**c.** [2 points] Let $h(x)$ be a differentiable function and define $H(x) = \int_0^x h(t)\,dt$. If $H(x)$ is always concave up, then $h(e^{-x})$ is __________ an increasing function.

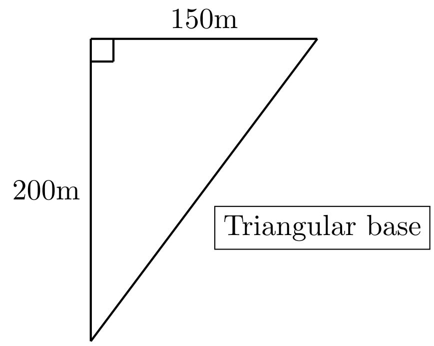
Always Sometimes $\boxed{\text{Never}}$

**d.** [2 points] Suppose a metal rod of density $\delta(x)$ lying along the $x$-axis from $x = -3$ to $x = 3$ has its center of mass at $x = 0$. Then the two halves of the rod on either side of $x = 0$ __________ have the same mass.

Always $\boxed{\text{Sometimes}}$ Never

**9**. [11 points] Advanced beings from another planet recently realized they left a stockpile of nanotechnology here on Earth. These tiny devices are stacked in the shape of a pyramid with a triangular base that is flat on the ground. Its base is a right triangle with perpendicular sides of length 150m and 200m. Two of the other three sides are also right triangles, and all three right angles meet at one corner at the base of the pile. The fourth side is a triangle whose sides are the hypotenuses of the other three triangles. (See diagrams below.)




The density of the contents of the pile at a height of $h$ meters above the ground is given by

$$\delta(h) = \frac{2}{\sqrt{1+h^2}} \text{ kg/m}^3.$$

For this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s$^2$.

**a**. [4 points] Write an expression in terms of $h$ that approximates the volume (in cubic meters) of a horizontal slice of thickness $\Delta h$ of the contents of the pile at a distance $h$ meters above the ground.

*Solution:* The cross section of the pile $h$ meters above the ground is a right triangle with legs of length $200-2h$ and $150-1.5h$ meters. (One can use e.g. linearity or similar triangles to find these lengths.) Therefore, the volume (in cubic meters) of the horizontal slice is approximately $\frac{1}{2}(200-2h)(150-1.5h)\Delta h$.

**b**. [3 points] Write, but do **not** to evaluate, an integral that gives the total mass of the pile of nanotechnology. Include units.

*Solution:* We mutiply the expression in (a) by the density to get an approximation for the mass of the slice and integrate that expression from 0 to 100 while replacing $\Delta h$ by $dh$ to find a total mass of

$$\int_0^{100} \frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\,dh \quad \text{kilograms.}$$

**c**. [4 points] The beings must return to Earth and collect the nanotech that they left behind. Suppose that the spaceship hovers 150 meters above the ground (directly above the pile) while recovering the nanotechnology. Write, but do **not** evaluate, an integral which gives the total work that must be done in order to lift all of the nanotech from the pile into the ship. Include units.

*Solution:* We multiply the expression in (a) by the density, acceleration due to gravity, and the distance required to lift it to a height of 150 m to estimate the work done to lift that slice up to the spaceship. Then we integrate this expression from 0 to 100 and replace $\Delta h$ by $dh$ to find that the total work done is

$$\int_0^{100} g\cdot(150-h)\cdot\frac{1}{2}(200-2h)(150-1.5h)\frac{2}{\sqrt{1+h^2}}\,dh \quad \text{Joules.}$$