Math 116 — Second Midterm — March 20, 2017

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	12	
3	5	
4	6	
5	10	

Problem	Points	Score
6	13	
7	10	
8	12	
9	12	
10	12	
Total	100	

1. [8 points] For each of the following, remember to show your work carefully.

a. [4 points] For which value(s) of k is the function

$$L(t) = e^{kt}$$

a solution to the differential equation $L'' - L' = 6L$?

If there are no such values, write "NONE" in the answer blank.

Solution: If $L(t) = e^{kt}$, then $L'(t) = ke^{kt}$ and therefore $L''(t) = k^2 e^{kt}$. So if $L(t)$ is a solution to the given differential equation, we find that

$$k^2 e^{kt} - ke^{kt} = 6e^{kt}.$$

Dividing by e^{kt} (which is never zero) gives

$$0 = k^2 - k - 6 = (k-3)(k+2).$$

So $L(t)$ is a solution to the differential equation when $k = -2$ and when $k = 3$.

Answer: $k =$ $-2, 3$

b. [4 points] For which value(s) of C is the function

$$y = (e^x - x - C)^{1/2}$$

a solution to the differential equation $2y\dfrac{dy}{dx} = y^2 + x$?

If there are no such values, write "NONE" in the answer blank.

Solution: Note that if $y = (e^x - x - C)^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}(e^x - x - C)^{-1/2}(e^x - 1)$$

and thus

$$2y\frac{dy}{dx} = (e^x - 1).$$

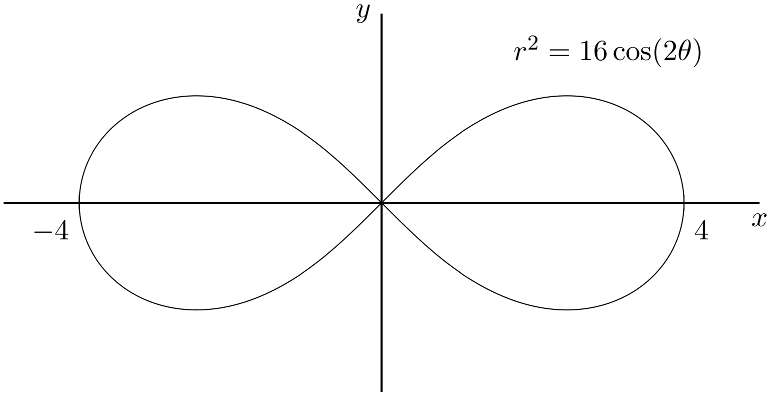
On the other hand,

$$y^2 + x = (e^x - x - C) + x = (e^x - C)$$

So if $y = (e^x - x - C)^{1/2}$ is a solution to the given differential eqution, then $e^x - 1 = e^x - C$ and thus $C = 1$.

Answer: $C =$ 1

2. [12 points] Chancelor was doodling in his coloring book one Sunday afternoon when he drew an infinity symbol, or lemniscate. The picture he drew is the polar curve $r^2 = 16\cos(2\theta)$, which is shown on the axes below. (The axes are measured in inches.)



a. [4 points] Chancelor decides to color the inside of the lemniscate red. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he has to fill in with red.

Solution: Notice that we are given a formula for r^2 instead of r. Using symmetry, we will calculate the area in one quarter of the lemniscate and multiply by 4. To do this we will integrate from $\theta = 0$ to $\theta = \alpha$ where α is the smallest positive number for which $16\cos(2\alpha) = 0$. This gives $\alpha = \pi/4$. Using the formula for area inside a polar curve we see that the area is equal to $4 \cdot \frac{1}{2}\int_0^{\pi/4} 16\cos(2\theta)\,d\theta$ square inches.

b. [4 points] He decides he wants to outline the right half (the portion to the right of the y-axis) of the lemniscate in blue. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total length, in inches, of the outline he must draw in blue.

Solution: The portion of the lemniscate on the right of the y-axis corresponds to $-\pi/4 < \theta < \pi/4$. Notice that $\cos(2\theta) > 0$ for these angles.

Implicitly differentiating $r^2 = 16\cos(2\theta)$ (or directly differentiating $r = 4(\cos(2\theta))^{1/2}$), we find that $\dfrac{dr}{d\theta} = \dfrac{-4\sin(2\theta)}{\sqrt{\cos(2\theta)}}$ so $\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{16\sin^2(2\theta)}{\cos(2\theta)}$.

Then using the arc length formula for polar coordinates we see that the length of the blue outline will be $\displaystyle\int_{-\pi/4}^{\pi/4}\sqrt{16\cos(2\theta) + \frac{16\sin^2(2\theta)}{\cos(2\theta)}}\,d\theta$ inches.

c. [4 points] Chancelor draws another picture of the same lemniscate, but this time also draws a picture of the circle $r = 2\sqrt{2}$. He would like to color the area that is inside the lemniscate but outside the circle purple. Write, but do **not** evaluate, an expression involving one or more integrals that gives the total area, in square inches, that he must fill in with purple.

Solution: The circle and lemniscate intersect on the right side of the y-axis when $16\cos(2\theta) = 8$ or $\cos(2\theta) = \frac{1}{2}$. This gives angles $\theta_1 = -\pi/6$ and $\theta_2 = \pi/6$.

We first find the area between the curves on the right side using the formula for the area between polar curves and then multiply the by 2.

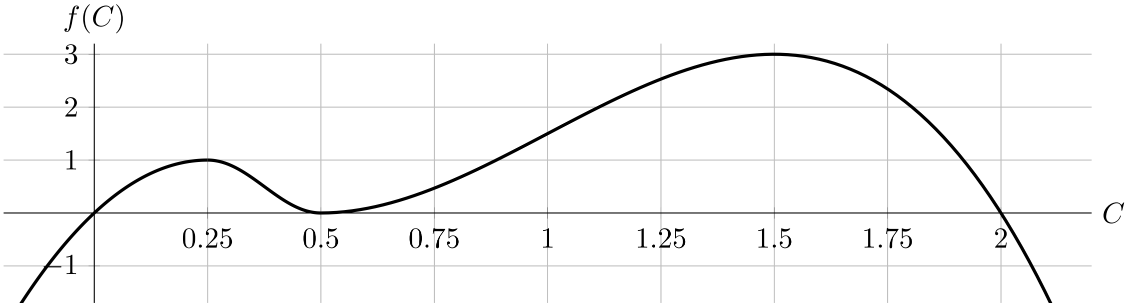
The resulting total area is $2 \cdot \frac{1}{2}\int_{-\pi/6}^{\pi/6}(16\cos(2\theta) - 8)\,d\theta$ square inches.

3. [5 points] Sasha and her friends are sipping lemonade on her boat when the boat begins to leak through a new hole in the bottom. Water begins to enter the boat at a constant rate of 1.5 gallons per minute. Immediately, they spring into action and begin to scoop the water out of the boat using lemonade pitchers that hold 0.25 gallons of water. That rate that the water is scooped, water, in scoops per minute, is proportional to the cube root of the volume of water currently in the boat, with constant of proportionality k. Let $W = W(t)$ be the volume of water in the boat, in gallons, t minutes after the leak begins. Write a differential equation that models $W(t)$, and give an appropriate initial condition.

Answer: Differential Equation: $\frac{dW}{dt} = 1.5 - 0.25kW^{1/3}$

Initial Condition: $W(0) = 0$

4. [6 points] Consider the differential equation $\frac{dC}{dt} = f(C)$ where $f(C)$ is the function graphed below.



a. [4 points] Identify all equilibrium solutions of this differential equation. Then indicate which of these equilibrium solutions are stable. Write your answers on the answer blanks provided.

Answer: All Equilibrium Solutions: $C = 0,\ C = 0.5,\ C = 2$

Stable Equilibrium Solutions: $C = 2$

b. [2 points] Suppose that a solution to this differential equation passes through a point with $C = 0.17$. For this solution, what will happen to the value of C as $t \to \infty$?

Solution: Note that $\frac{dC}{dt} > 0$ for $0.17 \le C < 0.5$, and that $C = 0.5$ is an equilibrium solution. So C will increase from 0.17 and approach 0.5. That is, $\lim_{t\to\infty} C = 0.5$.

5. [10 points] Let $f(x)$ and $g(x)$ be two functions that are differentiable on $(0, \infty)$ with continuous derivatives and which satisfy the following inequalities for all $x \geq 1$:

$$\frac{1}{x} \leq f(x) \leq \frac{1}{x^{1/2}} \qquad \text{and} \qquad \frac{1}{x^2} \leq g(x) \leq \frac{1}{x^{3/4}}.$$

For each of the following, determine whether the integral always, sometimes, or never converges. Indicate your answer by circling the one word that correctly fills the answer blank. No justification is necessary. No credit will be awarded for unclear markings.

a. [2 points] $\int_1^\infty \sqrt{f(x)}\, dx$ ____________ converges.

Always Sometimes [✓ Never]

b. [2 points] $\int_3^\infty 4000 g(x)\, dx$ ____________ converges.

Always [✓ Sometimes] Never

c. [2 points] $\int_1^\infty f(x) g(x)\, dx$ ____________ converges.

[✓ Always] Sometimes Never

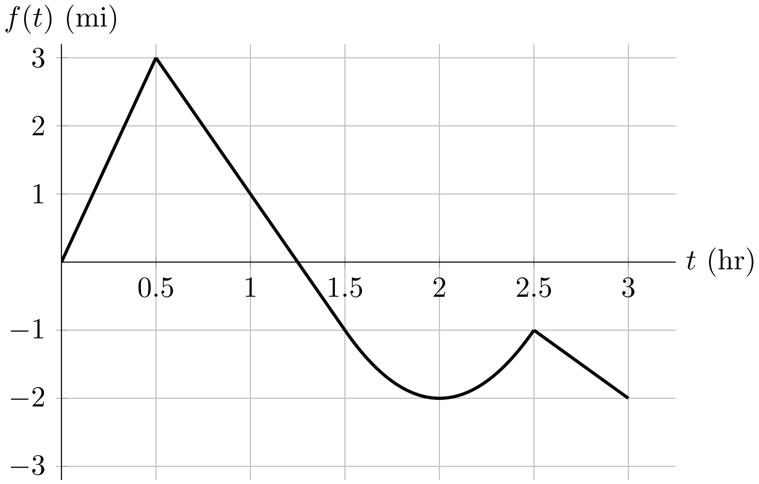
d. [2 points] $\int_5^\infty g'(x) e^{g(x)}\, dx$ ____________ converges.

[✓ Always] Sometimes Never

e. [2 points] $\int_1^\infty f'(x) \ln(f(x))\, dx$ ____________ converges.

[✓ Always] Sometimes Never

6. [13 points] Anderson and Glen decide to take a road trip starting from Venice Beach. They have no worries about getting anywhere quickly, as they enjoy each other's company, so they take a very inefficient route. Suppose that Venice Beach is located at $(0,0)$ and that Anderson and Glen's position (x, y) (measured in miles) t hours after leaving Venice Beach is given by a pair of parametric equations $x = f(t)$, $y = g(t)$. A graph of $f(t)$ and a formula for $g(t)$ are given below. Note that $f(t)$ is linear on the intervals $[0, 0.5]$, $[0.5, 1.5]$, and $[2.5, 3]$.



$$g(t) = -t^3 + 5t^2 - 3t$$

Note: For each of the following, your final answer should **not** involve the letters f and g.

a. [2 points] If their roadtrip last 3 hours, what are the $x-$ and $y-$ coordinates of their final destination?

Solution: Note that at time $t = 3$, we have $x = f(3) = -2$ and $y = g(3) = 9$. So the coordinates of their final destination are $(-2, 9)$.

b. [3 points] At what speed are they traveling 2 hours into their trip?

Solution: We have $\left.\frac{dx}{dt}\right|_{t=2} = f'(2) = 0$ and $\left.\frac{dy}{dt}\right|_{t=2} = g'(2) = 5.$

So their speed at time $t = 2$ is $\sqrt{0^2 + 5^2} = 5$ miles per hour.

c. [4 points] Write, but do not compute, an expression involving one or more integrals that gives the distance they traveled, in miles, in the first **half** hour of their trip.

Solution: On the interval $(0, 0.5)$, we see that $f(t) = 6t$, so on this interval, we have

$$f'(t) = 6 \quad \text{and} \quad g'(t) = -3t^2 + 10t - 3.$$

The parametric arc length formula then implies that the distance they travelled from $t = 0$ to $t = 0.5$ is $\displaystyle\int_0^{0.5} \sqrt{(6)^2 + (-3t^2 + 10t - 3)^2}\, dt$ miles.

d. [4 points] Write down a pair of parametric equations using the parameter s for the line tangent to their path at $t = 2.75$ hours.

Solution: Note that

$$f(2.75) = -1.5, \quad \left.\frac{df}{dt}\right|_{t=2.75} = -2, \quad g(2.75) = 8.765625, \quad \text{and} \quad \left.\frac{dg}{dt}\right|_{t=2.75} = 1.8125$$

There are many possible parametrizations. There is no need to have this match with the parameter t from earlier, so the answer below has the line passing through $(-1.5, 8.765625)$ at $s = 0$.

Answer: $x(s) =$ $-2s - 1.5$ and $y(s) =$ $1.8125s + 8.765625$

7. [10 points] Fatimah begins to make herself a cup of tea to soothe her sore throat. As soon as the water boils, her telephone rings and she answers it. While she is on the phone, the hot water begins to cool. The temperature of the hot water (in °C) is given by a function $H = H(t)$, where t is measured in minutes after the water boils. Suppose $H(t)$ satisfies the differential equation

$$\frac{dH}{dt} = k(H - 20)t^2$$

for some constant k.
(Note: This is **not** Newton's Law of Cooling. Her teapot has unusual thermal properties.)

Assume that the water boils at 100°C.

a. [1 point] What are the units of the constant k?

Solution: $1/\text{min}^3$

b. [6 points] Use separation of variables to find a formula for $H(t)$ by hand. Your formula may involve k, but should not involve any other unknown constants. Be sure to show your work.

Solution: Separating variables gives

$$\frac{dH}{H - 20} = kt^2\,dt.$$

Integrating we find,

$$\int \frac{dH}{H - 20} = \int kt^2\,dt$$
$$\ln|H - 20| = \frac{kt^3}{3} + C,$$

for some constant C.
Exponentiation of both sides of this equation gives

$$|H - 20| = e^{kt^3/3 + C} = Ae^{kt^3/3},$$

where $A = e^C$ is a positive real number. Then we have

$$H - 20 = Be^{kt^3/3}$$
$$H = 20 + Be^{kt^3/3}$$

where $B = \pm A$ is a nonzero real number. Note that $H = 20$ (corresponding to $B = 0$) is an equilibrium solution, so the general solution is $H = 20 + Be^{kt^3/3}$ where B is any real number.
Since the initial condition is $H(0) = 100$, we find that $100 = 20 + B$, so $B = 80$, and the particular solution is $H = 20 + 80e^{kt^3/3}$.

c. [3 points] Suppose that 3 minutes after boiling, the temperature of the water is 60°C. What is the value of k?

Solution: This new information tells us that $H(3) = 60$. Plugging this in to our solution above, we find that $60 = 20 + 80e^{k3^3/3} = 20 + 80e^{9k}$.
So,

$$80e^{9k} = 40$$
$$e^{9k} = 0.5$$
$$k = \ln(0.5)/9.$$

8. [12 points] For each of the questions below, circle all of the available correct answers.
Circle "NONE OF THESE" if none of the available choices are correct.
No justification is required. No credit will be awarded for unclear markings.

a. [3 points] The table below gives some values of a function $g(y)$.

y	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$g(y)$	1	2	4	6	5	1	2	3	5	4	2	1	3

Suppose $y = y(x)$ is a function of x, and consider the differential equation $y' = g(y)$ with initial condition $y(1) = 3$. Then the Euler's Method approximation of $y(2)$ when the step size is $\Delta x = 0.5$ is

i. 3 ii. 3.5 iii. 4 iv. 4.5 v. 5

vi. 5.5 vii. 6 viii. $\boxed{6.5}$ ix. 7 x. 7.5

xi. CANNOT BE DETERMINED FROM THE INFORMATION PROVIDED

XII. NONE OF THESE

b. [3 points] $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{4x} =$

i. $\boxed{e^{4/3}}$ ii. e^{12} iii. ∞ (DNE) iv. 0

v. 1 vi. $\ln(12)$ vii. $\ln(400/9)$ viii. NONE OF THESE

c. [3 points] Which of the following pairs of parametric equations trace out a curve that lies entirely on the right half of the unit circle centered at the origin?

i. $x = \cos^2(t)$ and $y = \sin^2(t)$ for $0 < t < \pi/2$

ii. $x = \cos(t)$ and $y = \sin(t)$ for $0 < t < 3\pi/4$

iii. $x = \sin(4t)$ and $y = \cos(4t)$ for $\pi/2 < t < \pi$

iv. $\boxed{x = \sqrt{1-t^2} \quad \text{and} \quad y = t \quad \text{for} \quad -1 < t < 1}$

v. NONE OF THESE

d. [3 points] Which of the following expressions give the total area inside the polar curve $r = 2\sin(\theta)$?

i. 4π ii. $\boxed{\pi}$ iii. $\boxed{2\displaystyle\int_0^{\pi}\sin^2\theta\,d\theta}$ iv. $2\displaystyle\int_0^{2\pi}\sin^2\theta\,d\theta$ v. NONE OF THESE

9. [12 points] Determine whether the following integrals converge or diverge.
If the integral converges, circle "converges", find its exact value (i.e. no decimal approximations), and write the exact value on the answer blank provided.
If the integral diverges, circle "diverges" and justify your answer.
In either case, **you must show all your work and indicate any theorems you used to conclude convergence or divergence of the integrals.** Any direct evaluation of integrals must be done **without using a calculator.**

a. [6 points] $\displaystyle\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$

[✓ **Diverges**] **Converges to** ________________

Solution: If the improper integral converges, we can rewrite it as a sum

$$\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx = \int_{-\infty}^{1} \frac{2}{(1+x^4)^{1/4}}\,dx + \int_{1}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx.$$

To show that the original improper integral diverges, is enough to show that the last improper integral above diverges.

We will use direct comparison. First, we note that for $x \geq 1$ $\dfrac{2}{(1+x^4)^{1/4}} \geq \dfrac{1}{x}$.

Additionally $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges as it is an integral of the form $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ with $p = 1 \leq 1$.

So by direct comparison, we see that the improper integral

$$\int_1^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$$

also diverges. Thus the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{2}{(1+x^4)^{1/4}}\,dx$ diverges.

b. [6 points] $\displaystyle\int_1^{e} \frac{x^3 - 3x^3\ln(x) - 1}{x(x^3-1)^2}\,dx$ *Hint:* $\displaystyle\frac{d}{dx}\left(\frac{\ln(x)}{x^3-1}\right) = \frac{x^3 - 3x^3\ln(x) - 1}{x(x^3-1)^2}$

Diverges [✓ **Converges**] **to** $\dfrac{1}{e^3-1} - \dfrac{1}{3}$

Solution:

$$\int_1^{e} \frac{x^3 - 3x^3\ln(x) - 1}{x(x^3-1)^2}\,dx = \lim_{b\to 1^+} \int_b^{e} \frac{x^3 - 3x^3\ln(x) - 1}{x(x^3-1)^2} = \lim_{b\to 1^+} \frac{\ln(x)}{x^3-1}\bigg|_b^e$$

$$= \lim_{b\to 1^+}\left(\frac{\ln(e)}{e^3-1} - \frac{\ln(b)}{b^3-1}\right) = \frac{1}{e^3-1} - \lim_{b\to 1^+}\frac{\ln(b)}{b^3-1}.$$

As this last limit has the indeterminate form $0/0$ we can apply L'Hopital's rule.

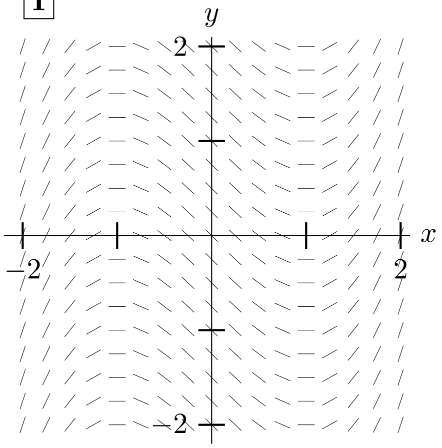
$$\lim_{b\to 1^+}\frac{\ln(b)}{b^3-1} = \lim_{b\to 1^+}\frac{1/b}{3b^2} \text{ (by L'Hopital's rule)}$$

$$= \lim_{b\to 1^+}\frac{1}{3b^3} = \frac{1}{3}.$$

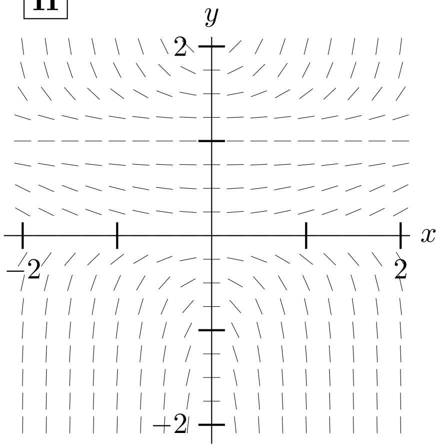
So we find that the improper integral $\displaystyle\int_1^{e} \frac{x^3 - 3x^3\ln(x) - 1}{x(x^3-1)^2}\,dx$ converges to $\dfrac{1}{e^3-1} - \dfrac{1}{3}$.

10. [12 points] Consider the three differential equations whose slope fields are shown in **I**, **II**, and **III** below.

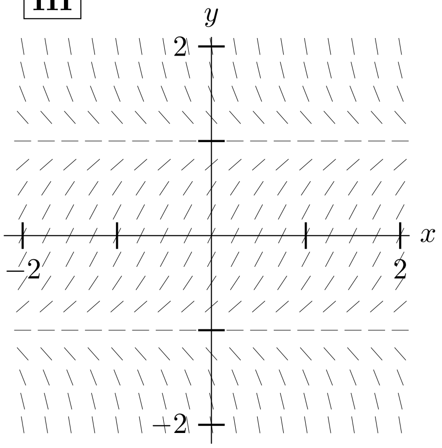
I



II



III



For each of the properties below, circle **all** of the slope fields for which the corresponding differential equation appears to satisfy that property. Circle "NONE OF THESE" if none of the differential equations satisfy the property. Explanation is not required. No credit will be awarded for ambiguous answers.

For each of the properties below, circle **all** of the slope fields for which the corresponding differential equation appears to satisfy that property. Circle "NONE OF THESE" if none of the differential equations satisfy the property. Explanation is not required. No credit will be awarded for ambiguous answers.

a. [2 points] $y = 1$ is a stable equilibrium solution of the differential equation.

I **II** [✓ **III**] NONE OF THESE

b. [2 points] $\frac{dy}{dx} < 0$ for $0 < x < 1$.

[✓ **I**] **II** **III** NONE OF THESE

c. [2 points] If we use Euler's method starting at the point $(1, 1.5)$ and use $\Delta x = 0.1$, the resulting estimate of $y(2)$ would be an underestimate of the actual value of $y(2)$.

[✓ **I**] [✓ **II**] [✓ **III**] NONE OF THESE

d. [2 points] The solution passing through the point $(0.5, 0.25)$ has $\lim_{x\to\infty} y = 1$.

I [✓ **II**] [✓ **III**] NONE OF THESE

e. [2 points] The differential equation can be written in the form $\frac{dy}{dx} = f(y)$ for some function f.

I **II** [✓ **III**] NONE OF THESE

f. [2 points] The approximate values arising from Euler's method starting at the point $(-1, -1)$ and using $\Delta x = 0.25$ lie on a straight line.

I **II** [✓ **III**] NONE OF THESE