Math 116 — Second Midterm — March 19, 2018

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	5	
2	5	
3	10	
4	8	
5	12	
6	12	

Problem	Points	Score
7	10	
8	7	
9	11	
10	10	
11	10	
Total	100	

1. [5 points] Let a_n be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = 4$, and let S_n be a sequence defined by $S_n = a_1 + a_2 + \cdots + a_n$. **No justification necessary.**

a. [2 points] Find the following limits. Write DNE if the limit does not exist or is ∞ or $-\infty$.

i. $\lim_{n\to\infty} a_n =$ **0**

ii. $\lim_{n\to\infty} S_n =$ **4**

b. [3 points]
Circle all statements which **must be true**.

i. a_n is increasing

ii. a_n is decreasing

iii. $\boxed{S_n \text{ is increasing}}$

iv. S_n is decreasing

v. $\boxed{S_n \text{ is bounded}}$

vi. None of these

2. [5 points] Calculate $\int_0^{\infty} \frac{2}{1+x^2}\,dx$. **Show all your work using correct notation.** Evaluation of integrals must be done **without a calculator.**

Solution:

$$\begin{aligned}
\int_0^{\infty} \frac{2}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^{b} \frac{2}{1+x^2}\,dx \\
&= \lim_{b\to\infty} 2\arctan(x)\Big|_0^b \\
&= \lim_{b\to\infty} 2\arctan(b) - 2\arctan(0) \\
&= 2\cdot\frac{\pi}{2} - 0 \\
&= \pi
\end{aligned}$$

3. [10 points] Consider the power series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{5^n(2n)!}(x-9)^n.$$

a. [1 point] What is the center of the interval of convergence of this power series?

Answer: $x = 9$

b. [5 points] What is the radius of convergence of this power series? Show your work.

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{((n+1)!)^2}{5^{n+1}(2n+2)!}\right)\left(\frac{5^n(2n)!}{(n!)^2}\right) = \lim_{n\to\infty}\frac{(n+1)^2}{5(2n+2)(2n+1)} = \frac{1}{20}.$$

Therefore $R = 1/(\frac{1}{20}) = 20$.

Answer: Radius of convergence = **20**

c. [4 points] A certain power series $\sum_{n=1}^{\infty} C_n(x-4)^n$ converges when $x = 1$ and diverges when $x = 13$. Which of the following could be the radius of converge of this series? Circle <u>all</u> possibilities from the list below.

0   1 $\boxed{3}$ $\boxed{7}$ $\boxed{9}$ 13 ∞ NONE OF THESE

4. [8 points] Consider the power series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{3^n n\sqrt{\ln(n)}}.$$

The radius of convergence of this power series is $R = 3$. Determine the interval of convergence for this power series and fully justify the convergence or divergence at the endpoints. **You may assume $R = 3$ without justification.**

Solution: Since the radius is 3 and the center is 1, we know that the series must converge for all values of x between $1-3=-2$ and $1+3=4$ and diverge for $x<-2$ and $x>4$. Now we check convergence at each of the endpoints.

When $x=-2$ we get the series

$$\sum_{n=2}^{\infty} \frac{(-3)^n}{3^n n\sqrt{\ln(n)}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln(n)}}.$$

This is an alternating series, and since $\dfrac{1}{n\sqrt{\ln(n)}}$ is decreasing and has limit 0, the series converges by the alternating series test.

When $x=4$ we get the series

$$\sum_{n=2}^{\infty} \frac{3^n}{3^n n\sqrt{\ln(n)}} = \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}.$$

Using the integral test, we consider the improper integral

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \int_{\ln(2)}^{\infty} \frac{1}{\sqrt{u}}\,du.$$

This integral diverges by the p-test ($p=1/2$). Therefore the interval of convergence is $[-2,4)$.

Answer: Interval of convergence = $[-2,4)$

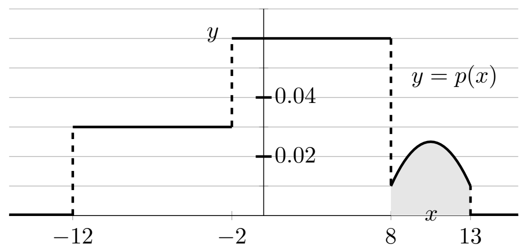
5. [12 points]

Yennifer's Introductory Thermodynamics of Muck course is supposed to start at **9:10 am**, but her instructor does not consistently start on time. Let $p(x)$ be the probability density function for the amount of time x, in minutes, between when the instructor is supposed to start the class and when they actually start class.

For parts **a.-c.**, you do not need to justify your answer.

Note that:

- $x = 0$ represents class starting at 9:10 am.
- A negative value of x represents starting class early.
- All of the nonzero portion of $p(x)$ is given in the graph below.
- The area of the shaded region is 0.1.



a. [2 points] Yennifer is coming from another class and therefore always arrives at 9:06, exactly 4 minutes before class is supposed to start. Find the probability that class starts before Yennifer arrives.

Answer: $8 \cdot 0.03 = 0.24 = 24\%$

b. [3 points] Which of the following statements is best supported by the equation $p(12) = 0.02$? Circle the **one** best answer.

i. The probability that the instructor will start class at 9:22 is 2%.

ii. The probability that the instructor will start class between 9:21 and 9:23 is about 2%.

iii. $\boxed{\text{The probability that the instructor will start class between 9:21 and 9:23 is about 4\%.}}$

iv. The probability that the instructor has started class by 9:22 is about 2%.

v. The probability that the instructor has started class by 9:22 is about 48%.

c. [3 points] Let $P(x)$ be the cumulative distribution function for $p(x)$. Which of the following could be the formula for $P(x)$ **on the interval** $-2 < x < 8$? Circle **all** answers that could be correct.

i. $P(x) = 0$

ii. $P(x) = 1$

iii. $P(x) = 0.06x$

iv. $P(x) = 0.06(x + 2)$

v. $\boxed{P(x) = 0.06(x + 2) + 0.3}$

vi. $P(x) = 0.1 - 0.06(x - 8)$

d. [4 points] Find the median value of x. Show your work, and write your answer in exact form.

Solution: We want to find x so that $\int_{-\infty}^{x} p(t)\,dt = 0.5$. The area under the curve for $x \le -2$ is $0.03 \cdot 10 = 0.3$, so we want x with $-2 \le x \le 8$ such that $0.3 + .06(x + 2) = 0.5$. Solving for x, we find $x = 0.2/0.06 - 2 = \frac{10}{3} - 2 = \frac{4}{3}$.

Answer: $4/3$

6. [12 points] Determine whether the following series converge or diverge.
Fully justify your answer. Show all work and indicate any convergence tests used.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: Since

$$\frac{n^2 + n\cos(n)}{\sqrt{n^8 - n + 1}} \approx \frac{1}{n^2},$$

we use the limit comparison test comparing with $\frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{n^2+n\cos(n)}{\sqrt{n^8-n+1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4 + n^3\cos(n)}{\sqrt{n^8 - n + 1}} = 1.$$

We can see this is true by using domination arguments: the numerator is dominated by n^4, while $\sqrt{n^8 - n + 1}$ is dominated by $\sqrt{n^8}$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test ($p = 2$), our original series converges by the LCT.

b. [6 points] $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: The terms in this series are not positive, but it is also not an alternating series. We will consider the series

$$\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right| = \sum_{n=0}^{\infty} \frac{|\sin(n)|}{e^n}. \tag{1}$$

Since $|\sin(n)| \leq 1$ we have

$$\frac{|\sin(n)|}{e^n} \leq \frac{1}{e^n}$$

for all $n \geq 0$. The larger series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges by the geometric series test since the common ratio $1/e$ is less than 1. (Note that there are many other ways to show that this series converges.) Therefore $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges by comparison. Since $\sum_{n=0}^{\infty} \left|\frac{\sin(n)}{e^n}\right|$ converges, our original series $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(n)}{e^n}$ converges absolutely, and, specifically, must itself converge (this is sometimes called the absolute convergence test).

7. [10 points] Consider the two sequences a_n and b_n defined by

$$a_n = \frac{1}{2^n} \qquad\qquad b_0 = 5, \quad b_n = 3b_{n-1} \text{ for all } n > 1.$$

Compute the following limits. If the sequence diverges, write DIVERGES.
No justification necessary.

a. [2 points] $\lim\limits_{n\to\infty} a_n$

Answer: $\lim\limits_{n\to\infty} a_n =$ $\mathbf{0}$

b. [2 points] $\lim\limits_{n\to\infty} \sum_{k=0}^{n} a_k$

Answer: $\lim\limits_{n\to\infty} \sum_{k=0}^{n} a_k =$ $\mathbf{2}$

c. [2 points] $\lim\limits_{n\to\infty} a_n b_n$

Answer: $\lim\limits_{n\to\infty} a_n b_n =$ ∞ **or** DIVERGES

d. [2 points] $\lim\limits_{n\to\infty} \frac{\ln(b_n)}{\ln(a_n)}$

Answer: $\lim\limits_{n\to\infty} \frac{\ln(b_n)}{\ln(a_n)} =$ $\frac{\ln(3)}{\ln(1/2)} \approx -1.585$

e. [2 points] $\lim\limits_{n\to\infty} \frac{1 - e^{3a_n}}{a_n}$

Solution: Since this is a limit of the form 0/0, we solve this by applying L'Hôpital's Rule.

Answer: $\lim\limits_{n\to\infty} \frac{1 - e^{3a_n}}{a_n} =$ -3

8. [7 points] Consider the integral

$$\int_1^\infty \frac{e^{rx}}{x}\,dx,$$

where r is a constant.

a. [3 points] Show that this integral converges for $r < 0$. **Show all work and indicate any convergence tests used.**

Solution: We know that $\frac{e^{rx}}{x} \le e^{rx}$ for all $x \ge 1$.

Further, when $r < 0$, we know that $\int_1^\infty e^{rx}\,dx$ converges by exponential decay test.

Therefore, by (direct) comparison test, $\int_1^\infty \frac{e^{rx}}{x}\,dx$ converges.

b. [4 points] Show that the integral diverges for $r \ge 0$. **Show all work and indicate any convergence tests used.**

Solution: Now, when $r \ge 1$, we know that $e^{rx} \ge 1$ for all $x \ge 1$, so $\frac{e^{rx}}{x} \ge \frac{1}{x}$.

$\int_1^\infty \frac{dx}{x}$ diverges by p-test with $p = 1$.

Therefore, by comparison, $\int_1^\infty \frac{e^{rx}}{x}\,dx$ diverges.

Alternative solution:

For $r > 0$, $\lim_{x\to\infty} \frac{e^{rx}}{x} = \infty$. Since the integrand approaches infinity, the integral diverges.

This still leaves the $r = 0$ case. In this case, $\frac{e^{rx}}{x} = \frac{1}{x}$, so the integral diverges by p-test with $p = 1$.

9. [11 points] Leight Vloss had trouble paying his rent so he started a cult. His followers believe that Leight receives holy messages from "The Great Consciousness" hiding in the internet. Each month Leight recruits D new followers and loses 20% of the followers he had in the previous month to disillusionment and other cults. That is, the number of followers Leight has after n months is described by the recursive formula

$$F_n = D + .8F_{n-1}$$

a. [4 points] Supposing that Leight has 0 followers the moment he gets the idea to start a cult, which is to say that $F_0 = 0$, compute the number of followers he has in the first three months. Your answer may be in terms of D.

Answer: $F_1 =$ D

$F_2 =$ $D + .8D = 1.8D$

$F_3 =$ $D + .8D + (.8)^2 D = 2.44D$

b. [4 points] Find a closed form expression for F_n, the number of followers Leight has after n months of channelling the spirit of the internet.

Solution: As we can see above, F_n is a finite geometric series with initial term D and ratio 0.8.

Answer: $F_n =$ $D \cdot \dfrac{1-.8^n}{1-.8}$

c. [3 points] Leight finds he needs the number of followers to tend to 1000 in the long run to ensure he can make rent each month. What's the fewest number of followers D that Leight Vloss needs to recruit each month to make sure he can pay rent?

Solution: Leight needs

$$\lim_{n\to\infty} F_n \geq 1000.$$

So

$$\lim_{n\to\infty} F_n = \lim_{n\to\infty} D\frac{1-.8^n}{1-.8} = \frac{D}{.2} \implies D \geq 200.$$

Answer: **200**

10. [10 points] Suppose $Q(x)$ is the cumulative distribution function (cdf) for a variable x, such that

$$Q(x) = \begin{cases} a & \text{for } x \leq 0 \\ b - e^{-cx} & \text{for } x > 0, \end{cases}$$

and the median value of x is 2.

a. [2 points] Let $q(x)$ be the probability density function for x. Write a formula for $q(x)$, assuming $q(0) = 0$.

Answer:

Solution: $Q(x)$ is an antiderivative of $q(x)$, so we can differentiate $Q(x)$ to get

$$q(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ ce^{-cx} & \text{for } x > 0 \end{cases}$$

b. [4 points] Set up, but <u>do not evaluate</u>, an expression involving one or more integrals that represents the mean value of x. Your answer may contain a, b, or c, but should <u>not</u> contain any function names (such at Q or q).

Answer: $\int_0^\infty cxe^{-cx}\,dx$

c. [4 points] Find the values of a, b, and c. Justify your answers, and write them in exact form. Remember that the <u>median</u> value of x is 2.

Solution: We know $\lim_{x\to-\infty} Q(x) = a = 0$ and $\lim_{x\to\infty} Q(x) = b = 1$. Finally, we have

$$\begin{aligned} Q(2) &= 1/2 \\ 1 - e^{2c} &= 1/2 \\ e^{-2c} &= 1/2 \\ -2c &= \ln(1/2) \\ c &= -\ln(1/2)/2 = \ln(2)/2 \end{aligned}$$

Answer: $a =$ **0** $b =$ **1** $c =$ $\ln(2)/2$

11. [10 points] You work for a temp agency. Today you fill in for Russ Weterson, doing important work for the city. On Mr. Weterson's desk you find the following problems with a note: "Russ, the Mayor needs these problems done yesterday. -Brontel"

Suppose $f(x)$ and $g(x)$ are positive, continuous, decreasing functions such that

1. $\int_1^\infty f(x)\,dx$ converges, and
2. $0 \le g(x) \le 9$ for all real numbers x.

Determine whether the following expressions must converge, must diverge, or whether convergence cannot be determined. **No justification required.**

a. [2 points] $\displaystyle\int_1^\infty \frac{1}{f(x)}\,dx$

CONVERGES $\boxed{\text{DIVERGES}}$ CANNOT BE DETERMINED

b. [2 points] $\displaystyle\sum_{n=1}^\infty f(n)$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT BE DETERMINED

c. [2 points] $\displaystyle\int_1^\infty f(x)g(x)\,dx$

$\boxed{\text{CONVERGES}}$ DIVERGES CANNOT BE DETERMINED

d. [2 points] $\displaystyle\sum_{n=1}^\infty f(n)^{g(n)}$

CONVERGES DIVERGES $\boxed{\text{CANNOT BE DETERMINED}}$

e. [2 points] $\displaystyle\int_1^\infty g(x)\,dx$

CONVERGES DIVERGES $\boxed{\text{CANNOT BE DETERMINED}}$