Math 116 — Final Exam — April 19, 2018

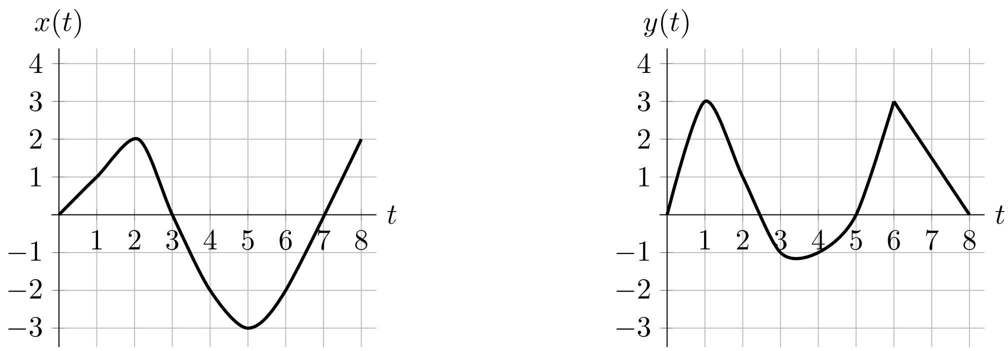
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 15 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	9	
3	10	
4	11	
5	10	
6	11	

Problem	Points	Score
7	6	
8	8	
9	7	
10	8	
11	6	
12	6	
Total	100	

1. [8 points] Once again, Giuseppe Li has a nightmare. This time he is roaming the plains in a covered wagon. His stepfather, Gerd Hömf, plotted the x- and y-coordinates of Giuseppe's dream wagon measured in astral miles as a function of dream days t. The positive y-direction is north, and the positive x-direction is east.



a. [2 points] At what time(s) t was Giuseppe's wagon moving directly north? Directly south? If there are no such times, write "NONE".

Answer: Directly north at $t =$ 5

Answer: Directly south at $t =$ 2

b. [2 points] How far, in astral miles, from where it began was Giuseppe's wagon after 8 dream days?

Answer: **2 astral miles**

c. [2 points] What was Giuseppe's approximate speed, in astral miles per dream days, at time $t = 7$?

Solution: speed at $t = 7$ is given by

$$\sqrt{x'(7)^2 + y'(7)^2} = \sqrt{2^2 + (-1.5)^2} = \frac{5}{2}$$

Answer: 2.5 **astral miles per dream day.**

d. [2 points] Approximate the total distance, in astral miles, that Giuseppe's wagon traveled during the first 2 dream days.

Solution: On the first day Giuseppe travels from $(0,0)$ to $(1,3)$ and on the second day from $(1,3)$ to $(2,1)$ along an approximately straight path each day. Hence the total distance is approximately

$$\sqrt{(1-0)^2 + (3-0)^2} + \sqrt{(2-1)^2 + (1-3)^2} = \sqrt{10} + \sqrt{5} \approx 5.398.$$

Answer: $\sqrt{10}+\sqrt{5} \approx 5.398$ **astral miles.**

2. [9 points] For a class project, Yennifer is studying the accumulation of dead leaves on the ground in a particular region on Nichols Arboretum.

a. [4 points] She finds that the dead leaves accumulate at a constant rate of 6 grams per square centimeter per year. At the same time, the leaves on the ground decompose at a continuous rate of 80 percent per year. Write a differential equation for the total quantity Q of dead leaves, in grams per square centimeter, at time t, in years.

Answer: $\frac{dQ}{dt} = 6 - .8Q$

b. [5 points] Yennifer finds that if she covers the ground in purified muck, then the total quantity P, in grams, of dead leaves per square centimeter satisfies the differential equation

$$\frac{dP}{dt} = (P-3)\cos(2\pi t)$$

In addition, when she first applies the muck (at $t = 0$), the ground is covered with 1 gram per square centimeter of leaves.

Use separation of variables to find a formula for $P(t)$. Show your work.

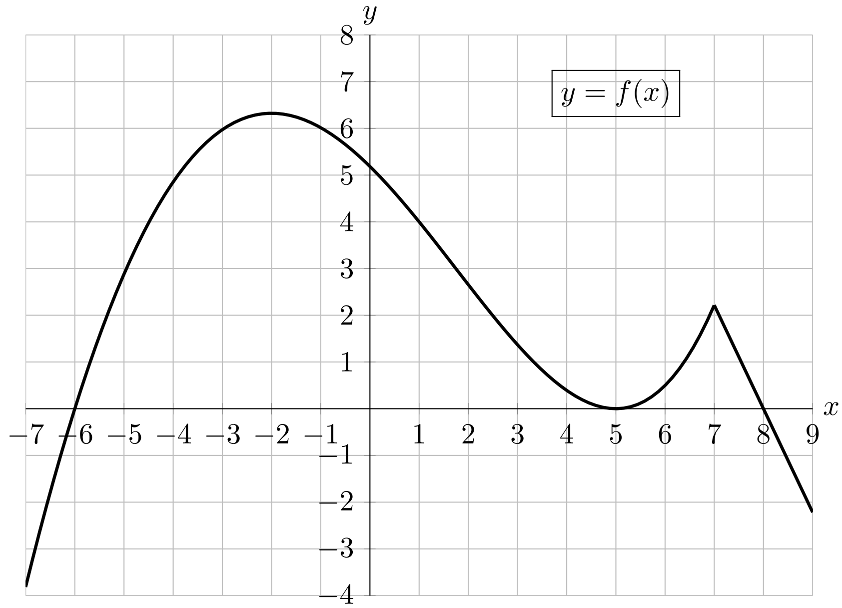
Solution: Separating variables gives

$$\int \frac{1}{P-3}\,dP = \int \cos(2\pi t)\,dt$$
$$\ln|P-3| = \frac{1}{2\pi}\sin(2\pi t) + C_0$$
$$P = C_1 e^{\frac{1}{2\pi}\sin(2\pi t)} + 3.$$

Using the initial condition $P(0) = 1$ we find $1 = C_1 + 3$, hence $C_1 = -2$.

Answer: $P(t) = 3 - 2e^{\frac{1}{2\pi}\sin(2\pi t)}$

3. [10 points] Consider the function $f(x)$ graphed below.



a. [3 points] Let $F(x) = \int_0^x f(t)\,dt$. Find the x-coordinates of all local extrema of $F(x)$ and classify them as local maxima or local minima. Write "NONE" if there are none.

Answer: Local maxima at $x =$ <u>8</u>

Answer: Local minima at $x =$ <u>-6</u>

b. [3 points] Let $G(x) = \int_{3x}^{x^2} f(t)\,dt$. Compute $G'(-1)$.

Solution: Using the second fundamental theorem of calculus we compute

$$G'(x) = f(x^2)\cdot 2x - f(3x)\cdot 3.$$

Therefore,

$$G'(-1) = -2f(1) - 3f(-3) = -26$$

Answer: $G'(-1) =$ <u>-26</u>

c. [2 points] Which approximation method is guaranteed to underestimate $\int_{-4}^{0} f(x)\,dx$?

MID $\boxed{\text{TRAP}}$ LEFT RIGHT NONE OF THESE

d. [2 points] Which approximation method is guaranteed to overestimate $\int_{-1}^{5} f(x)\,dx$?

MID TRAP $\boxed{\text{LEFT}}$ RIGHT NONE OF THESE

4. [11 points]

a. [6 points] Determine whether the following series converges absolutely, converges conditionally, or diverges, and give a complete argument justifying your answer.

$$\sum_{n=1}^{\infty}(-1)^n \sin\left(\tfrac{1}{n}\right)$$

Converges absolutely $\quad$ $\boxed{\textbf{Converges conditionally}}$ $\quad$ **Diverges**

Justification:

Solution: This series converges by the alternating series test, which applies, since $\sin(\frac{1}{n})$ is a positive decreasing sequence that converges to zero.

It does not converge absolutely since for $n \geq 1$

$$\frac{1}{2n} \leq \sin\left(\tfrac{1}{n}\right).$$

We know the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$ diverges by p-test with $p = 1$. Then by the comparison test, so must $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right) = \sum_{n=1}^{\infty}\left|(-1)^n \sin\left(\tfrac{1}{n}\right)\right|$.

Alternatively, we can use the Limit Comparison Test. Since

$$\lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{x\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{y\to 0}\frac{\sin(y)}{y} = 1 < 0,$$

we know that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right)$ must either both converge or both diverge. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series, which we know diverges, $\displaystyle\sum_{n=1}^{\infty}\sin\left(\tfrac{1}{n}\right)$ must diverge as well.

b. [5 points] Compute the value of the following improper integral. **Show all your work using correct notation.** Evaluation of integrals must be done **without a calculator.**

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$$

Solution:
First we change to limit notation, then use u-substitution with $u = 1 + e^x$.

$$\begin{aligned}\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx\\ &= \lim_{b\to\infty}\int_2^b \frac{1}{u^2}\,du\\ &= \lim_{b\to\infty} -\frac{1}{u}\Big|_2^b\\ &= \lim_{b\to\infty}\frac{-1}{b} - \frac{-1}{2} = \frac{1}{2}\end{aligned}$$

Alternatively, first compute the antiderivative using u-substitution.

$$\int \frac{e^x}{(1+e^x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{1+e^x}.$$

Thus,

$$\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{(1+e^x)^2}\,dx = \lim_{b\to\infty} -\frac{1}{1+e^b} + \frac{1}{2} = \frac{1}{2}.$$

5. [10 points] Compute the **exact value** of each of the following. You do not need to show work.

a. [2 points] Find the radius of convergence R of $\displaystyle\sum_{n=1}^{\infty} \frac{5(x-1)^n}{3^n}$.

Solution: By the geometric series test this series converges if

$$\frac{|x-1|}{3} < 1$$

which implies that the radius of convergence is $R = 3$.

Answer: $R =$ **3**

b. [2 points] $\displaystyle\sum_{n=1}^{100} e^n$

Answer: $\displaystyle\sum_{n=1}^{100} e^n =$ $\frac{e^{101}-e}{e-1}$

c. [2 points] $\displaystyle\lim_{n\to\infty} \int_{-\infty}^{n} p(t)\,dt$, where $p(t)$ is a probability density function.

Answer: $\displaystyle\lim_{n\to\infty} \int_{-\infty}^{n} p(t)\,dt =$ **1**

d. [2 points] Find the function $f(x)$ satisfying $\displaystyle\int x^3 e^x\,dx = x^3 e^x - \int f(x)\,dx$

Solution: Taking the derivative of both sides we find that

$$x^3 e^x = 3x^2 e^x + x^3 e^x - f(x).$$

Therefore $f(x) = 3x^2 e^x$.

Answer: $f(x) =$ $3x^2 e^x$

e. [2 points] $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!}$

Solution: After rewriting the series as

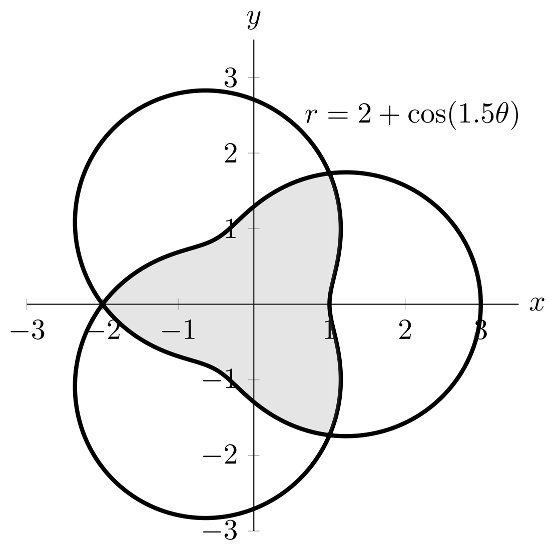
$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!},$$

we recognize this as $\cos(2)$.

Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} =$ $\cos(2)$

6. [11 points]

Leight Vloss has instructed his Star Children to run laps on the Trail of Atonement. The Trail of Atonement is best described as the polar curve $r = 2 + \cos(1.5\theta)$ where r is measured in kilometers. An aerial view of the trail is illustrated below.



a. [4 points] Leight stands on a pedestal in the center of the trail (at the origin). What is the furthest distance, in km, a Star Child gets from Leight on the Trail of Atonement? List all angles θ in $[0, 4\pi)$ where this distance r is achieved.

Solution: Since the maximum value of $\cos(1.5\theta)$ is 1 it follows that r is at most 3. To find the angles where this maximum r is achieved, we set

$$\begin{aligned} 2 + \cos(1.5\theta) &= 3 \\ \cos(1.5\theta) &= 1 \\ \frac{3}{2}\theta &= 0, 2\pi, 4\pi, \ldots \end{aligned}$$

So the values in $[0, 4\pi)$ are $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$.

Alternatively, since we're looking for angles where r achieves its maximum, we can set the derivative of $r = 2 + \cos(1.5\theta)$ equal to 0. Since $r'(\theta) = -1.5\sin(1.5\theta)$ we have that $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$.

Answer: Greatest distance: **3 km**

Answer: $\theta =$ $\theta = 0, \frac{4\pi}{3}, \frac{8\pi}{3}$

b. [3 points] Write an integral in terms of θ which represents the total length, in km, of the Trail of Atonement.

Solution: Since the Trail of Atonement is described by $r = f(\theta) = 2 + \cos(1.5\theta)$ with $0 \le \theta < 4\pi$ we use the polar arc length formula to write

$$\int_0^{4\pi} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_0^{4\pi} \sqrt{(2+\cos(1.5\theta))^2 + (-1.5\sin(1.5\theta))^2}\, d\theta.$$

Alternatively, if we use $x = (2 + \cos(1.5\theta)\sin(\theta)$ and $y = (2 + \cos(1.5\theta))\sin(\theta)$, and then the parametric formula for arc length, we get

$$\int_0^{4\pi} \sqrt{\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta$$

where

$$\frac{dy}{d\theta} = -1.5\sin(1.5\theta)\sin(\theta) + (2 + \cos(1.5\theta))\cos(\theta)$$

and

$$\frac{dx}{d\theta} = -1.5\sin(1.5\theta)\cos(\theta) - (2 + \cos(1.5\theta))\sin(\theta)$$

c. [4 points] The shaded innermost region of the trail is called the Sacred Heart. Write an expression involving one or more integrals which represents the area, in km^2, of the Sacred Heart.

Solution: The corners of the Sacred heart occur when $r = 2$ as we can see from the leftmost corner. The first two solutions of

$$2 = 2 + \cos(1.5\theta)$$

are $\theta = \frac{\pi}{3}, \pi$. These angles describe 1/3 of the Sacred Heart, so the total area is

$$3\int_{\frac{\pi}{3}}^{\pi} \frac{f(\theta)^2}{2}\, d\theta = 3\int_{\frac{\pi}{3}}^{\pi} \frac{(2+\cos(1.5\theta))^2}{2}\, d\theta.$$

7. [6 points] The function $r(t)$, defined for all real numbers t, gives the position of a particle moving along the unit circle,

$$r(t) = (\cos(t - t^3), \sin(t - t^3)).$$

a. [3 points] Find all values of t where the particle stops moving.

Solution: The particle stops moving when its speed is zero. The speed is given by

$$\sqrt{(-\sin(t - t^3)(1 - 3t^2))^2 + (\cos(t - t^3)(1 - 3t^2))^2} = |1 - 3t^2|.$$

Therefore the speed is zero at $t = \pm\frac{1}{\sqrt{3}}$.

Answer: $t =$ $\pm\frac{1}{\sqrt{3}}$

b. [3 points] For which values of t is the particle moving counterclockwise?

Solution: The parametric function $r(t)$ moves counterclockwise precisely when $f(t) = t - t^3$ is increasing, which is the same as $f'(t) > 0$. Since $f'(t) = 1 - 3t^2$, this happens for t in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

Answer: $-\frac{1}{\sqrt{3}} < t < \frac{1}{\sqrt{3}}$

8. [8 points] Let $f(x) = xe^{-x^2}$.

a. [4 points] Find the first four nonzero terms of the Taylor series for $f(x)$ centered at $x = 0$.

Answer: $x - x^3 + \frac{x^5}{2} - \frac{x^7}{6}$

b. [2 points] Find the value of $f^{(18)}(0)$.

Solution: Since $f(x)$ is an odd function, there are no odd powers of x in the Taylor series expansion of $f(x)$ centered at $x = 0$. Since the coefficient of x^{18} is $\frac{f^{(18)}(0)}{18!}$, it follows that $f^{(18)}(0) = 0$.

Answer: $f^{(18)}(0) =$ **0**

c. [2 points] Compute the limit

$$\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3}.$$

Solution: Using the Taylor polynomial found in the first part we have

$$\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3} = \lim_{x\to 0} \frac{-x^3 + \frac{x^5}{2} - \frac{x^7}{6}}{5x^3} = \lim_{x\to 0} -\frac{1}{5} + \frac{x^2}{10} - \frac{x^4}{30} = -\frac{1}{5}.$$

Note that this answer is exact and not an approximation, since all later terms in the Taylor series have x^n for $n > 7$, and so will go to 0 even when divided by x^3.

Answer: $\lim_{x\to 0} \frac{xe^{-x^2} - x}{5x^3} =$ $-\frac{1}{5}$

9. [7 points] Brontel Muskell claims his phone's screen was broken by a penny falling from the top of the New Toledo television tower and now he wants money from the insurance company. Giuseppe Li, the actuary in charge of the case, does not believe Brontel's claim.
The differential equation modelling the velocity v of a falling object subject to air resistance is

$$v' = g - rv^2 \tag{1}$$

where v is downward velocity, in m/s, g is acceleration due to gravity, and r is a positive constant depending on the shape and size of the falling object.

a. [3 points] The positive equilibrium solution $v = v_T$ to (1) is called the *terminal velocity*. Find v_T in terms of g and r. Is v_T stable or unstable?

Solution: Solving $0 = g - rv^2$ for v we find $v = \sqrt{\frac{g}{r}}$. This is a stable equilibrium.

Answer: $v_T =$ $\sqrt{\frac{g}{r}}$

v_T is (circle one) $\boxed{\text{STABLE}}$ UNSTABLE

b. [4 points] Giuseppe speaks with the insurance company's lead scientist, Tammy Toppel, who conducts some experiments and concludes that a penny needs to be moving at least 30 m/s to break the phone screen.

Use the company's estimates that $r = \frac{1}{40}$ for a falling penny and $g = 10$ to compute the value of v_T. Then use this value and your answer about the stability of v_T to help Dr. Toppel write a response to Brontel's claim.

Answer: Terminal velocity: 20 **m/s**

This finding (circle one) SUPPORTS $\boxed{\text{CONTRADICTS}}$

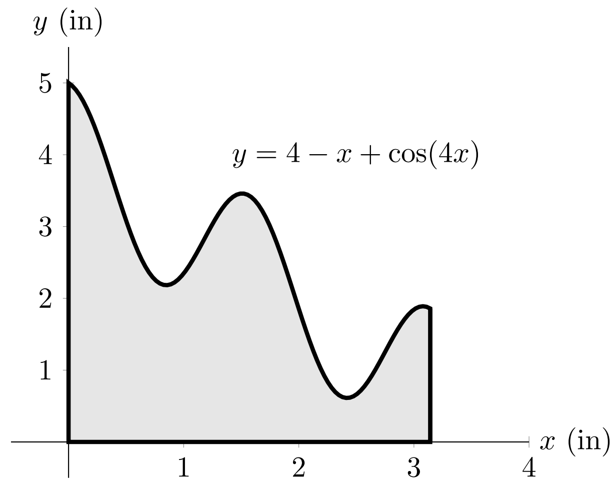
Brontel's claim because... *(briefly give reasoning below)*

Solution: Since $v_T = 20$ meters per second is a stable equilibrium one would expect a falling penny from a great height to be moving at very nearly this speed. Since this is slower than the speed required for a penny to break Brontel's phone screen, it seems he's lying.

10. [8 points] Recently Debra McQueath was thinking about all the great things she used to make at `Print.juice` by revolving regions around the y-axis. Those were the good days, weren't they?

a. [4 points]

There was that one time she designed the Juice Titan™ formed by rotating the region in the first quadrant bounded by $x = \pi$ and $y = 4 - x + \cos(4x)$ around the y-axis. The density $\delta(x)$ of the plastic was a function of the distance from the center of the juicer, although Debra cannot quite remember what it was. Help Debra write an integral that represents the total mass of the Juice Titan™. Your integral may include the density function $\delta(x)$.

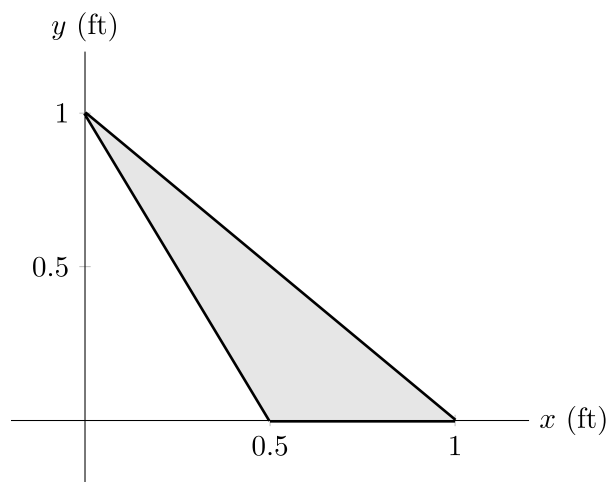


Solution: Use the shell method.

Answer: $\displaystyle\int_0^{\pi} 2\pi x\delta(x)(4 - x + \cos(4x))\,dx$

b. [4 points]

On Debra's last day at `Print.juice` her team made her a commemorative hat containing a hollow chamber filled with juice by rotating the region bounded by $y = 0$, $y = 1 - x$ and $y = 1 - 2x$ around the y-axis. The juice-filled hat still sits on her kitchen table; she sometimes wonders how much juice is in the hat. Write an integral that represents the total volume of juice in the hat. **Note:** juice fills the solid formed by rotating the shaded region.



Solution: Use the washer method.

Answer: $\displaystyle\int_0^{1} \pi\left((1-y)^2 - ((1-y)/2)^2\right)dy$

11. [6 points] The polynomial $P_3(x) = 2 - 3(x+e)^2 + 5(x+e)^3$ is the third-degree Taylor polynomial approximating the function $g(x)$ for x near $-e$. Find the following values. Write "NI" if there is not enough information.

$g'(-e) =$ 0 $\qquad g(-e) =$ 2 $\qquad g'''(-e) =$ 30

$P_3^{(4)}(-e) =$ 0 $\qquad g(0) =$ NI $\qquad P_3(0) =$ $2 - 3e^2 + 5e^3$

12. [6 points] Match the differential equations to their corresponding slope fields.

i. $y' = x^2 + y^2$ **D**

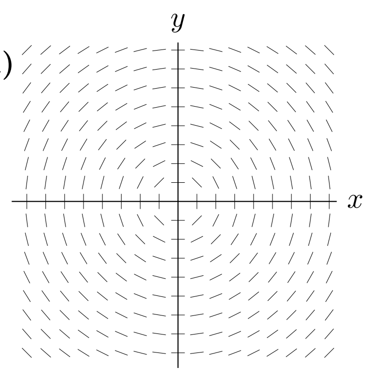
ii. $y' = \frac{y}{x}$ **B**

iii. $y' = -\frac{x}{y}$ **A**

iv. $y' = x(y^2 - 1)$ **E**

v. $y' = x(1 - y^2)$ **F**

vi. $y' = \frac{3x^2+1}{2y}$ **C**

(A)


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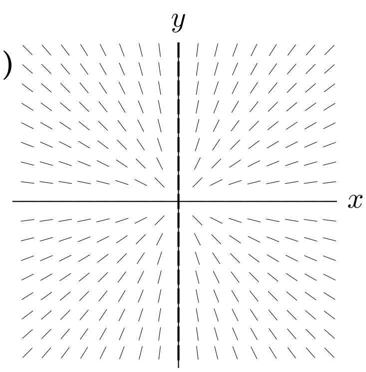

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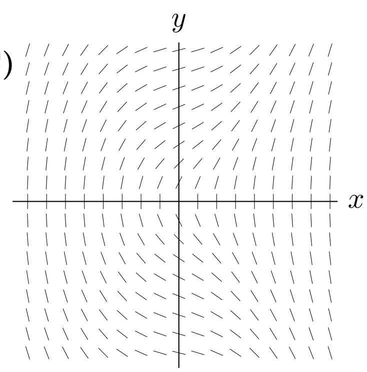

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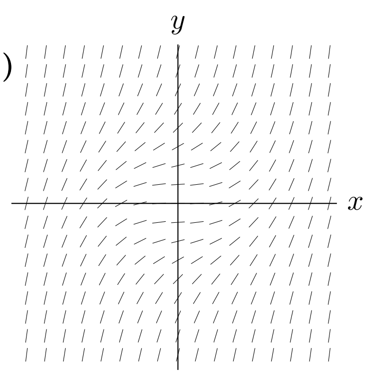

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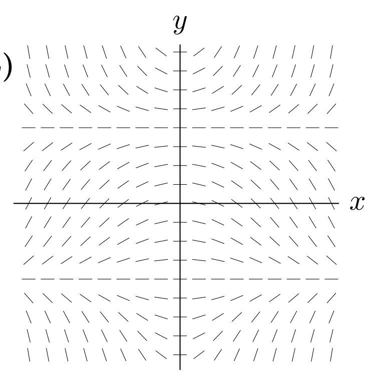

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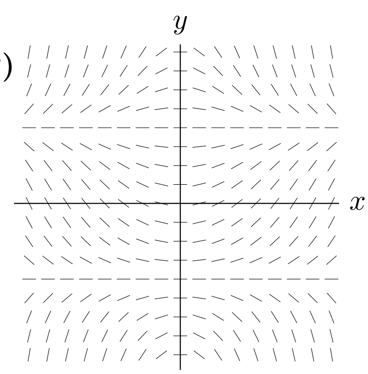

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$