Math 116 — First Midterm — February 11, 2019

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	16	
3	11	
4	11	
5	9	

Problem	Points	Score
6	12	
7	12	
8	12	
9	6	
Total	100	

1. [11 points] Let $f(x)$ be a differentiable function with continuous derivative, and $F(x) = \int_2^x f(t)\,dt$. Some values of the functions $f(x)$ and $F(x)$ are shown below:

x	-1	0	1	2	3
$f(x)$	7	4	0.25	9	8
$F(x)$	8	9	0.5	0	1

Compute the exact numerical values of the following integrals. If it is not possible to do so based on the information provided, write "NOT POSSIBLE" and clearly indicate why it is not possible. Show your work.

a. [3 points] $\int_0^1 x f'(x)\,dx$

Solution: By parts.

$$\begin{aligned}\int_0^1 x f'(x)\,dx &= x f(x)\Big|_0^1 - \int_0^1 f(x)\,dx \\ &= x f(x)\Big|_0^1 - F(x)\Big|_0^1 \\ &= f(1) - F(1) + F(0) = 0.25 - 0.5 + 9 = 8.75\end{aligned}$$

Answer: $8.75 = 35/4$

b. [3 points] $\int_{-1}^0 \sqrt[3]{x} f'(x^{4/3})\,dx$

Solution: Use substitution with $u = x^{4/3}$.

$$\int_{-1}^0 \sqrt[3]{x} f'(x^{4/3})\,dx = \frac{3}{4}\int_1^0 f'(u)\,du = \frac{3}{4}\,f(u)|_1^0 = \frac{3}{4}(4 - 0.25) = \frac{45}{16} = 2.8125$$

Answer: $45/16 = 2.8125$

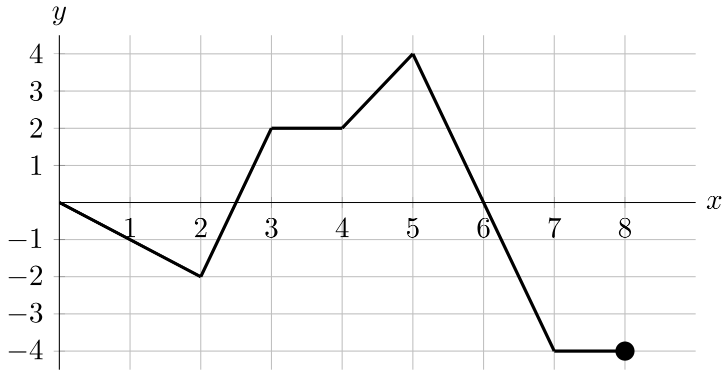
c. [5 points] $\int_1^2 \frac{f(x)}{(F(x))^2 - 1}\,dx$

Solution: Substitute $u = F(x)$ and then use partial fractions.

$$\begin{aligned}\int_1^2 \frac{f(x)}{(F(x))^2 - 1}\,dx = \int_{F(1)}^{F(2)} \frac{1}{u^2 - 1}\,du &= \int_{0.5}^0 \frac{1/2}{u-1} - \frac{1/2}{u+1}\,du \\ &= \frac{1}{2}\ln|u-1|\Big|_{0.5}^0 - \frac{1}{2}\ln|u+1|\Big|_{0.5}^0 \\ &= \frac{1}{2}(\ln 1 - \ln 0.5) - \frac{1}{2}(\ln 1 - \ln 1.5) \\ &= \frac{1}{2}(\ln 1.5 - \ln 0.5) = \frac{1}{2}\ln 3 = \ln\sqrt{3}\end{aligned}$$

Answer: $(\ln 1.5 - \ln 0.5)/2 = \ln 3/2 = \ln\sqrt{3}$

2. [16 points] Part of the graph of $g(x)$, a piecewise-linear odd function defined on $[-8, 8]$, is given below.

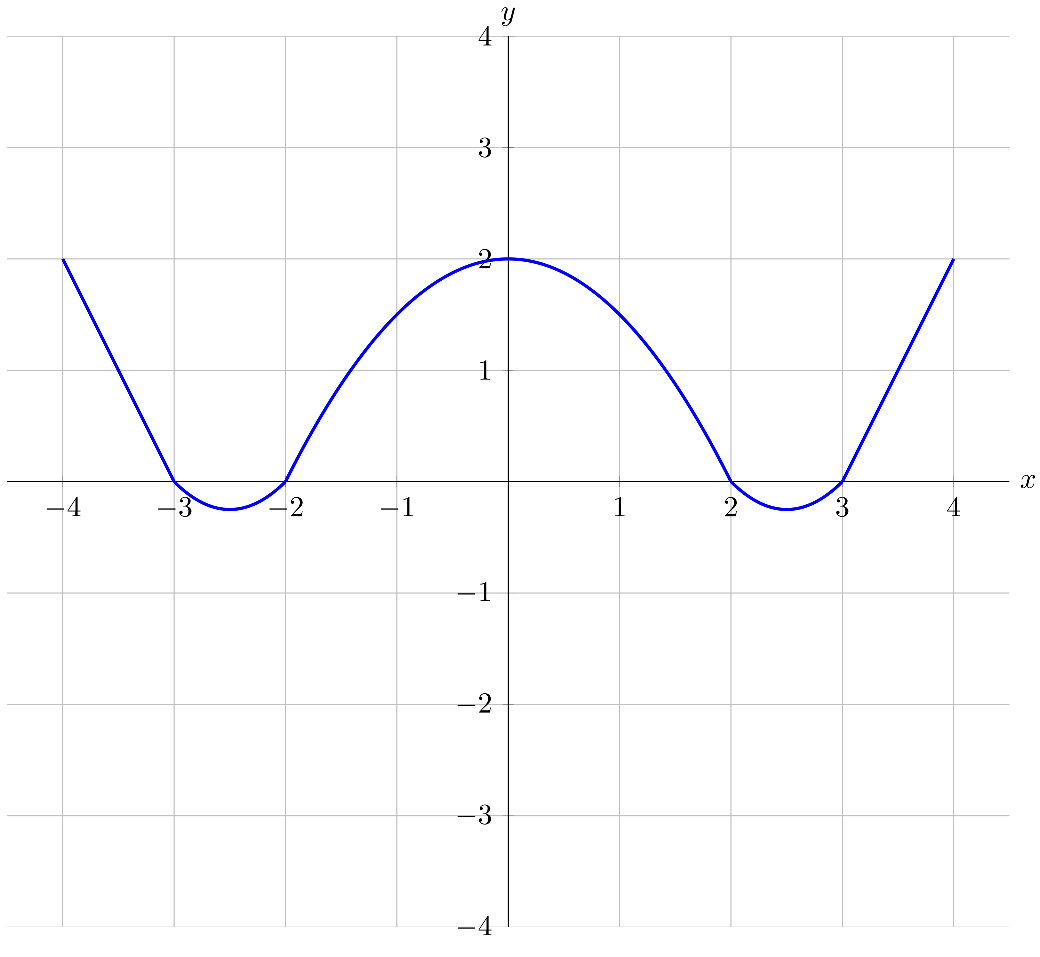


a. [6 points] Let $A(x) = \int_4^{2x} g(t)\,dt$. Find the following values. If the value does not exist, write "DNE". You do not need to show work, but partial credit may be awarded for correct work.

(i) $A(4) =$ **-1** (ii) $A(1) =$ **-2** (iii) $A'(2.5) =$ **8**

b. [10 points] Let $G(x) = \int_2^x g(t)\,dt$ for $-4 \le x \le 4$. Carefully sketch the graph of $G(x)$ below. Make sure your sketch clearly displays:

- the values of $G(x)$ at integer values of x;
- where $G(x)$ is and is not differentiable;
- where $G(x)$ is increasing or decreasing;
- the concavity of $G(x)$.



3. [11 points] The parts of this problem are not related.

a. [6 points] Suppose $f(x)$ is a positive function, defined for all real numbers x, with continuous first derivative. For each part below, circle "True" if the statement is **always** true and circle "False" otherwise. No justification is necessary.

$\int_0^3 xf(x^2)\,dx = \frac{1}{2}\int_0^3 f(u)\,du$ True [✓ False]

$\int_0^3 xf(x^2)\,dx = \int_0^3 sf(s^2)\,ds$ [✓ True] False

$\int xf(x^2)\,dx = x \cdot \int f(x^2)\,dx$ True [✓ False]

$\int xf(x^2)\,dx = x \cdot \int f(x^2)\,dx + f(x^2) \cdot \int x\,dx$ True [✓ False]

$\int xf(x^2)\,dx = \int x\,dx \cdot \int f(x^2)\,dx$ True [✓ False]

$\int xf(x^2)\,dx = \frac{x^2}{2}f'(x^2) - \int x^3 f'(x^2)\,dx$ True [✓ False]

b. [2 points] Suppose $G(x)$ and $H(x)$ are continuous antiderivatives of an even function $g(x)$ and $G(1) > H(1)$. Which of the following must be true?

i. [✓ $G(-1)$ is definitely greater than $H(-1)$.]

ii. $G(-1)$ is definitely not greater than $H(-1)$.

iii. None of these.

c. [3 points] A region bounded entirely by the graph of the function $y = \arctan(x)$, the y-axis, and the line $y = \frac{\pi}{4}$ is rotated around the x-axis. Which of the following integrals represents the volume of the resulting solid? Choose the <u>one</u> best answer.

i. $\pi\int_0^1 \left(\frac{\pi}{4} - \arctan(x)\right)^2 dx$

ii. [✓ $\pi\int_0^1 \left(\frac{\pi}{4}\right)^2 - (\arctan(x))^2\,dx$]

iii. $\pi\int_0^{\pi/4} 1 - (\arctan(x))^2\,dx$

iv. $\pi\int_0^{\pi/4} (\tan(y))^2 - 1\,dy$

v. $\pi\int_0^{\pi/4} (\tan(y) - 1)^2\,dy$

vi. $\pi\int_0^1 \left(\tan(y) - \frac{\pi}{4}\right)^2 dy$

vii. $\pi\int_0^1 (\tan(y))^2 - \left(\frac{\pi}{4}\right)^2 dy$

viii. None of these

4. [11 points] A polar vortex arrives in a college town at midnight, causing the temperature to drop. Consider the following:

- Let t be the time, in hours, after the polar vortex arrives.
- Let $r(t)$ be the rate, in degrees Fahrenheit per hour, at which the temperature is changing at time t.
- At first the temperature drops quickly, but as time passes, it drops less quickly.
- When the polar vortex first arrives, the temperature is 22° F.

Some values of $r(t)$ are given in the table below.

t	0	2	4	6	8
$r(t)$	-8	-5	-3	-2	-1

a. [2 points] Which of the following expressions must be the average rate of change of the temperature between midnight and 6 am? Circle all correct answers.

$$\boxed{\frac{1}{6}\int_0^6 r(t)\,dt} \qquad \frac{1}{6}\int_0^6 r'(t)\,dt \qquad \frac{\int_0^6 r(t)\,dt - 22}{6}$$

$$\frac{r(0)+r(2)+r(4)+r(6)}{4} \qquad \frac{r(6)-r(0)}{6}$$

None of these

b. [3 points] Write an expression involving a definite integral that represents the temperature, in degrees Fahrenheit, at 8 am.

Answer: $22 + \int_0^8 r(t)\,dt$

c. [6 points] At 8:00 am, Alexis is walking to class, and says, "I can't believe they didn't cancel classes! It must be 20 below out here!" Assuming Alexis means that the temperature is less than or equal to −20° F, is Alexis correct? Circle your answer, and show work or explain your reasoning below.

Alexis is correct [✓ Alexis is incorrect] Cannot be determined

Justification:

Solution: We are told that $r(t)$ is increasing ("At first the temperature drops quickly, but as time passes, it drops less quickly"), so LEFT will be an underestimate, which is what we need in this case. (We are not told the concavity of $r(t)$, but graphing the points we have make it seem likely that the function will be concave down, which would make TRAP an underestimate as well.)
Using LEFT(4) to approximate $\int_0^8 r(t)\,dt + 22$ gives us
$\text{LEFT}(4) = 2(-8 + -5 + -3 + -2) = -36$, so $22 + -36 = -14$

Since LEFT(4) is an underestimate, the actual temperature cannot be less than -14, and therefore Alexis is incorrect.

5. [9 points] Before calculators existed, it was a difficult task for scientists and engineers to compute values of special functions like logarithm. In this problem, we will use simple arithmetic operations to approximate the value of $\ln 2$.
Note that for $x > 0$,

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

a. [3 points] Approximate the integral $\int_1^2 \frac{1}{t}\,dt$ using LEFT(4). Write out each term in your sum.

Solution:

$$\begin{aligned}\text{LEFT}(4) &= \frac{2-1}{4}\left(\frac{1}{1}+\frac{1}{1+1/4}+\frac{1}{1+2/4}+\frac{1}{1+3/4}\right)\\ &= \frac{1}{4}(1+\frac{4}{5}+\frac{4}{6}+\frac{4}{7})\\ &= \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}.\end{aligned}$$

Answer: $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$

b. [3 points] Which of the following are equal to the LEFT(n) approximations for $\int_1^2 \frac{1}{t}\,dt$? Circle the one best answer.

i. $\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{i}$

ii. $\frac{1}{n}\sum_{i=0}^{n}\frac{1}{1+i/n}$

iii. $\boxed{\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{1+i/n}}$

iv. $\frac{1}{n}\sum_{i=1}^{n}\frac{1}{i}$

v. $\frac{1}{n}\sum_{i=n}^{n}\frac{1}{1+i/n}$

vi. $\frac{1}{n}\sum_{i=0}^{n}\frac{1}{1+1/n}$

c. [3 points] How many subintervals would be needed so that a scientist who lived before calculators were invented would be certain that the resulting left-hand Riemann sum approximates $\ln(2)$ to within 0.01? Justify your answer.

Solution:

$$|\text{LEFT}(n) - \text{RIGHT}(n)| \le \left|\frac{1}{1}-\frac{1}{2}\right|\cdot\frac{2-1}{n} = \frac{1}{2}\cdot\frac{1}{n} = \frac{1}{2n}$$

Since $1/2n \le 0.01$ if and only if $n \ge 50$, at least 50 subintervals would be needed.

Answer: **(at least)** 50

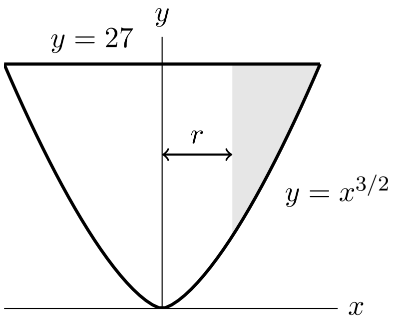
6. [12 points] Ryan Rabbitt is making a smoothie with his new electric drink mixer. Mathematically, the container of the mixer has a shape that can be modeled as the surface obtained by rotating the region in the first quadrant bounded by the curves $y = 27$ and $y = x^{3/2}$ about the y-axis, where all lengths are measured in centimeters.

a. [7 points] Write, but do not evaluate, two integrals representing the total volume, in cm^3, the mixer can hold: one with respect to x, and one with respect to y.

Answer (with respect to x): $\int_0^9 2\pi x \left(27 - x^{3/2}\right)\, dx$

Answer (with respect to y): $\int_0^{27} \pi \left(y^{2/3}\right)^2\, dy$

b. [5 points] Ryan adds 1600 cubic centimeters of liquid to his mixer. The container spins around the y-axis at a very high speed, causing the liquid to move away from the center of the container. The result is the solid made by rotating the shaded region around the y-axis in the diagram below. Note that this means that there is an empty space inside the liquid that has the shape of a cylinder.



Let r be the radius of this cylinder of empty space. Set up an equation involving one or more integrals that you would use to solve to find the value of r. **Do** **not** **solve for** r**.**

Solution:

$$\int_r^9 2\pi x \left(27 - x^{3/2}\right)\, dx = 1600,$$

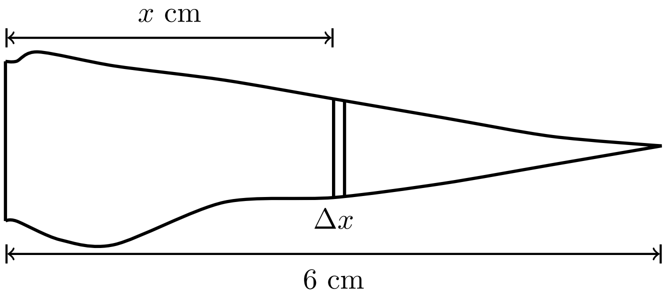
or

$$\int_{r^{3/2}}^{27} \pi \left(y^{2/3}\right)^2\, dy - \pi r^2 (27 - r^{3/2}) = 1600.$$

(There are other equations that would also work.)

Answer: ____________________

7. [12 points] Hannah Haire has a carrot that is 6 cm long. Lying on its side, it looks like the diagram below, and cross-sections perpendicular to the x-axis are circles. The density of the carrot also varies with x.



Given a distance x cm from the large end of the carrot, let $f(x)$ model the diameter, in cm, of the circular cross-section and $\delta(x)$ the density of the carrot, in g/cm^3.

a. [4 points] Write an expression that gives the approximate mass, in grams, of a slice of the carrot that is Δx cm thick and x cm from the large end of the carrot. (Assume here that Δx is small but positive.) Your expression should not involve any integrals, but may include $f(x)$ and $\delta(x)$.

Answer: $\pi\left(\frac{f(x)}{2}\right)^2\delta(x)\Delta x$

b. [3 points] Write an expression involving one or more integrals that gives the total mass of the carrot. Your answer may include $f(x)$ and $\delta(x)$.

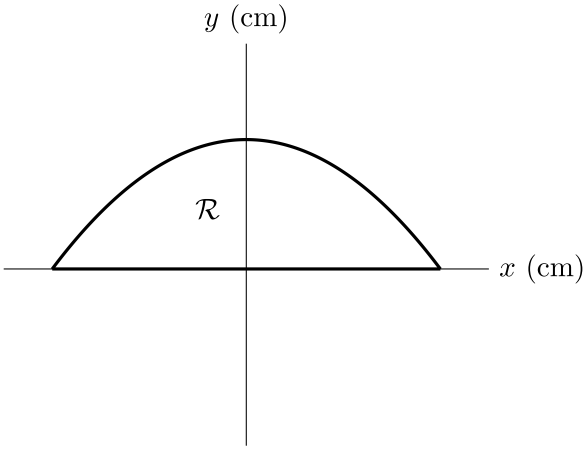
Answer: $\pi\int_0^6\left(\frac{f(x)}{2}\right)^2\delta(x)\,dx$

c. [5 points] Below is a table with some values of $f(x)$ and $\delta(x)$. Use MID(3) to estimate the mass, in grams, of the carrot. Write out every term in your sum.

x	0	1	2	3	4	5	6
$f(x)$	3.4	3.8	2.6	2.1	1.4	0.6	0
$\delta(x)$	1.54	1.52	1.48	1.44	1.42	1.39	1.32

Answer: $2\pi((\frac{3.8}{2})^2\cdot 1.52+(\frac{2.1}{2})^2 1.44+(\frac{0.6}{2})^2 1.39)\approx 45.3$

8. [12 points] Consider the region $\mathcal{R}$ bounded by the curve $x^2 + 3y = 4$ and the x-axis.



a. [4 points] Write an expression involving one or more integrals that gives the perimeter, in cm, of $\mathcal{R}$. You do not need to evaluate the integral.

Solution: The parabola intersects the x-axis at $x = \pm 2$. Moreover, we find $y = \frac{1}{3}(4 - x^2)$ so $\frac{dy}{dx} = \frac{2}{3}x$. We plug this into the arc length formula and add the length along the bottom.

Answer: $\int_{-2}^{2} \sqrt{1 + \frac{4}{9}x^2}\, dx + 4$

b. [4 points] Write an expression involving one or more integrals that gives the volume, in cm^3, of the solid formed by rotating $\mathcal{R}$ about the line $x = -4$.

Solution: Using shell method:

$$\int_{-2}^{2} 2\pi(x - (-4))\frac{1}{3}(4 - x^2)\, dx \text{ cm}^3 = \frac{2\pi}{3}\int_{-2}^{2} (x + 4)(4 - x^2)\, dx \text{ cm}^3.$$

Using washer method:

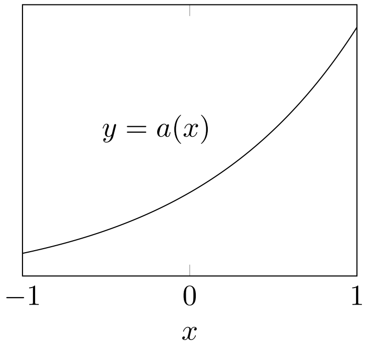
$$\int_{0}^{4/3} \pi\left(\sqrt{4 - 3y} - (-4)\right)^2 - \pi\left(-\sqrt{4 - 3y} - (-4)\right)^2\, dy$$

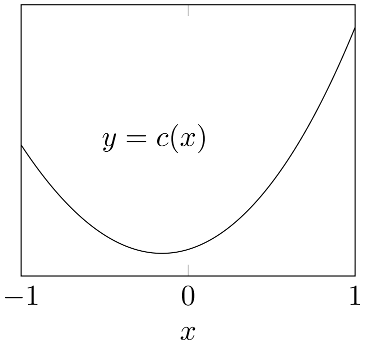
Answer: **(see above)**

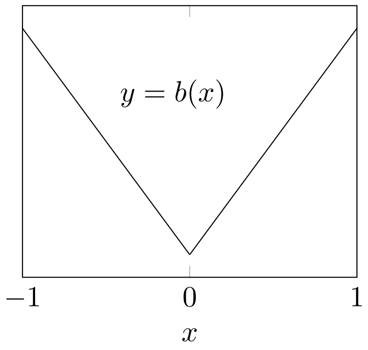
c. [4 points] Write, but do not evaluate, an expression involving one or more integrals that gives the mass, in grams, of a thin plate in the shape of the region R that has mass density given by $\delta(x) = x + 2$ g/cm^2.

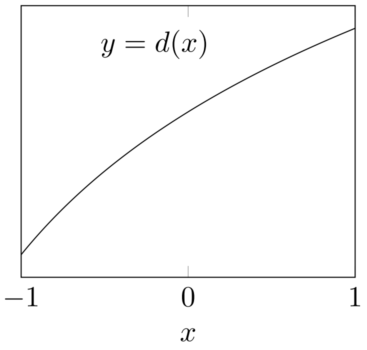
Answer: $\int_{-2}^{2} \frac{1}{3}(4 - x^2)(x + 2)\, dx$

9. [6 points] Below are the graphs of four functions. Note that the vertical scales are not given and may not be the same.









We have calculated LEFT(6), RIGHT(6), TRAP(6), AND MID(6) for the definite integral of each of three of these functions on the interval $[-1, 1]$. These estimates are listed below. For each one, circle the corresponding function.

(i)

LEFT(6)	1.21875
RIGHT(6)	1.58496
MID(6)	1.40185
TRAP(6)	1.29890

Function: $a(x)$ $c(x)$ $b(x)$ [✓ $d(x)$]

(ii)

LEFT(6)	5.1552
RIGHT(6)	8.0374
MID(6)	5.0882
TRAP(6)	6.5963

Function: $a(x)$ [✓ $c(x)$] $b(x)$ $d(x)$

(iii)

LEFT(6)	42.00
RIGHT(6)	42.00
MID(6)	42.00
TRAP(6)	42.00

Function: $a(x)$ $c(x)$ [✓ $b(x)$] $d(x)$