Math 116 — Second Midterm — March 25, 2019

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. Do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	8	
3	10	
4	9	
5	12	

Problem	Points	Score
6	9	
7	10	
8	12	
9	9	
10	9	
Total	100	

1. [12 points] The parts of this problem are unrelated. You do not need to justify your answers.

a. [6 points] For each of the following sequences, defined for $n \geq 1$, decide if it is bounded, if it is increasing or decreasing, and whether it converges, and circle your answers. If it converges, find the limit.

i. $b_n = \dfrac{2n + e^{-n}}{5n}$

Bounded Increasing **Decreasing**

Diverges **Converges to** **2/5**

ii. $c_n = \sin(n)$

Bounded Increasing Decreasing

Diverges Converges to ______

iii. $d_n = \displaystyle\sum_{k=1}^{n} \frac{3}{k}$

Bounded **Increasing** Decreasing

Diverges Converges to ______

b. [3 points] Write the following series using sigma notation:

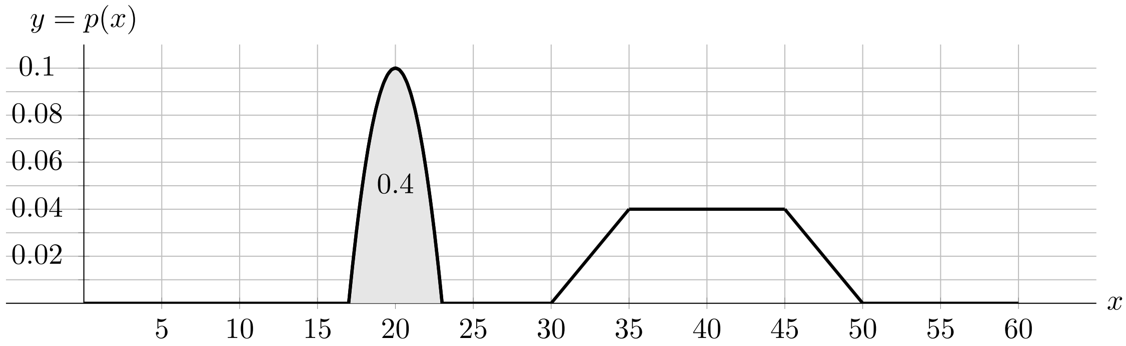
$$2^3(x-e)^4 + 3^3(x-e)^6 + 4^3(x-e)^8 + \ldots$$

Answer: $\displaystyle\sum_{n=2}^{\infty} n^3(x-e)^{2n} = \sum_{n=1}^{\infty} (n+1)^3(x-e)^{2(n+1)}$

c. [3 points] Suppose the power series $\displaystyle\sum_{n=0}^{\infty} C_n(x-2)^n$ converges at $x = 5$ and diverges at $x = 9$. Which of the following could be the radius of convergence R? Circle all correct answers.

$R = 0$ $R = 2$ $\boxed{R = 3}$ $\boxed{R = 4}$ $\boxed{R = 7}$ $R = 8$ $R = 10$

2. [8 points] Casey sometimes rides her bicycle in to school, and other times rides the bus. Let $p(x)$ be the probability density function for x, the time, in minutes, it takes Casey to get to school. The graph of $p(x)$ is given below. The area of the shaded region is 0.4.



a. [1 point] When she rides her bicycle, it always takes her significantly less time than when she takes the bus. What fraction of the days Casey goes to school does she ride her bike in?

Answer: **0.4 or 40%**

b. [2 points] If Casey always leaves her apartment exactly 40 minutes before class is supposed to start, what is the probability that she is late (assuming class starts on time)?

Answer: **0.30 or 30%**

c. [2 points] What is the median amount of time it takes Casey to get to school?

Answer: **35 min**

d. [3 points] Which of the following is the <u>one best</u> interpretation of the equation $p(19) = 0.06$?

i. About 6% of the time, it takes Casey exactly 19 minutes to get to school.

ii. About 6% of the time, it takes Casey at most 19 minutes to get to school.

iii. About 6% of the time, it takes Casey between 18 and 20 minutes to get to school.

iv. $\boxed{\text{About 12\% of the time, it takes Casey between 18 and 20 minutes to get to school.}}$

v. About 12% of the time, it takes Casey at most 18 to 20 minutes to get to school.

3. [10 points] For the following parts, show all of your work and indicate any theorems you used to conclude convergence or divergence of the integrals. Any direct evaluation of integrals must be done without using a calculator.

a. [5 points] Determine whether the improper integral $\int_0^\infty \frac{1}{x^3+1}\,dx$ converges or diverges.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

Solution: Write

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \int_0^1 \frac{1}{x^3+1}\,dx + \int_1^\infty \frac{1}{x^3+1}\,dx,$$

and observe that the first integral on the right-hand side is proper. For the second integral, note that

$$\frac{1}{x^3+1} \leq \frac{1}{x^3}$$

for $1 \leq x < \infty$, and the integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges by p-test with $p = 3 > 1$. We conclude that $\int_1^\infty \frac{1}{x^3}\,dx$ also converges by comparison test, therefore the original integral $\int_1^\infty \frac{1}{x^3}\,dx$ also converges.

b. [5 points] Give full justification of the following equation. Write out all steps by hand, including any integration techniques used.

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = 1$$

Solution:

$$\begin{aligned}
\int_1^\infty \frac{\ln(x)}{x^2}\,dx &= \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx \\
&= \lim_{b\to\infty} \left(-\frac{\ln(x)}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx \right), \qquad \text{by IBP with } u = \ln(x), dv = \frac{1}{x^2}dx, \\
&= \lim_{b\to\infty} \left(-\frac{\ln(b)}{b} - \frac{1}{b} + 1 \right) \\
&= -\lim_{b\to\infty} \left(\frac{\ln(b)}{b} \right) + 1 \\
&\overset{H}{=} -\lim_{b\to\infty} \left(\frac{1/b}{1} \right) + 1 \\
&= 1.
\end{aligned}$$

4. [9 points] Michigan Atomic and Thermonuclear Headquarter (M.A.T.H.) recently discovered a new chemical element X, which is radioactive with a half-life of 1 day. Currently, the M.A.T.H. lab is scheduled to synthesize k grams of X everyday at noon.
Let m_n be the mass (in grams) of X the M.A.T.H. lab has in possession at noon on the nth day of production, *immediately after* the new batch is produced; for example, $m_1 = k$.

a. [2 points] Calculate m_2 and m_3.

Answer: $m_2 = \dfrac{k}{2} + k$

Answer: $m_3 = \dfrac{k}{4} + \dfrac{k}{2} + k$

b. [4 points] Find a closed form expression for m_n.

Answer: $m_n = \dfrac{k\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 2k\left(1 - \left(\frac{1}{2^n}\right)\right)$

c. [3 points] The M.A.T.H lab plans to conduct an experiment on the element X which requires having 10 grams of X at once. At this production level, for what values of k can the experiment be carried out at some point in the future?

Solution: Since

$$\lim_{n\to\infty} m_n = 2k,$$

and we need $m_n \geq 10$ for some n, it follows that $2k > 10$ and therefore $k > 5$.

Answer: $k > 5$

5. [12 points] Determine whether each of the following series converges, conditionally converges, or diverges. Fully justify your answer. Include any convergence tests used.

a. [6 points] $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$

Circle one: $\boxed{\textbf{Absolutely convergent}}$ **Conditionally convergent** **Divergent**

Justification:

Solution: Since $ne^{-n} < 1$ for all $n \geq 1$,

$$ne^{-2n} = (ne^{-n})e^{-n} \leq e^{-n}$$

and the series $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ converges since it is a geometric series with common ratio e^{-1} whose magnitude is strictly less than 1, so $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ converges by comparison test.

Since $|ne^{-2n}| = ne^{-2n}$, the series $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ is absolutely convergent.

b. [6 points] $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$

Circle one: **Absolutely convergent** $\boxed{\textbf{Conditionally convergent}}$ **Divergent**

Justification:

Solution: Since $\frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ is decreasing and has limit zero, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ converges by the alternating series test.

On the other hand, since

$$\frac{\sqrt{n}+2}{\sqrt{n^3+1}} \geq \frac{\sqrt{n}}{\sqrt{n^3+n^3}} = \frac{1}{2n},$$

and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by p-test with $p = 1$, we conclude by comparison test that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{\sqrt{n^3+1}}$ diverges. Therefore, the original series is only conditionally convergent.

6. [9 points] For each of the following questions, circle all answers that must be correct.

a. [3 points] Circle all true statements. The integral $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx$

i. diverges because $\dfrac{1}{\sqrt{x}+x^2} > \dfrac{1}{2\sqrt{x}}$ for $0 < x < 1$.

ii. diverges because $\dfrac{1}{\sqrt{x}+x^2} > \dfrac{1}{2x^2}$ for $1 < x < \infty$.

iii. converges because $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}+x^2} = 0$.

iv. converges by p-test with $p = 2$.

v. $\boxed{\text{None of these.}}$

b. [3 points] Consider a geometric series with n^{th} partial sum S_n, where $\displaystyle\lim_{n\to\infty} S_n = \frac{5}{1-0.3}$. Which of the following statements must be true?

i. $\boxed{\text{This geometric series must converge.}}$

ii. The first term of this geometric series must be 0.3.

iii. The common ratio of this geometric series must be 0.3.

iv. This geometric series may or may not converge; it cannot be determined.

v. None of these.

c. [3 points] The series $1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{3^2} + \frac{1}{2^2} - \frac{1}{3^3} + \frac{1}{2^3} - \dots$

i. converges by the Alternating Series Test.

ii. diverges because the Alternating Series Test does not apply.

iii. neither converges nor diverges.

iv. converges because it is a geometric series with common ratio of magnitude less than 1.

v. converges because the terms converge to 0.

vi. $\boxed{\text{None of these.}}$

7. [10 points]

a. [5 points] Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^2(2n)!}{2^n(n!)^2} x^{2n}$.

Solution: Compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{(n+1)^2(2(n+1))!}{2^{n+1}((n+1)!)^2}|x|^{2(n+1)}}{\frac{n^2(2n)!}{2^n(n!)^2}|x|^{2n}} &= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{(2n+2)!}{(2n)!} \frac{2^n}{2^{n+1}} \frac{(n!)^2}{((n+1)!)^2} \frac{|x|^{2n+2}}{|x|^{2n}} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{2n^2}|x|^2 \\
&= 2|x|^2.
\end{aligned}$$

By ratio test, the power series converges for

$$2|x|^2 < 1 \iff |x|^2 < \frac{1}{2} \iff |x| < \frac{1}{\sqrt{2}}.$$

Answer: $\frac{1}{\sqrt{2}}$

b. [5 points] You do not need to justify your answers below.
Suppose C_n is a sequence such that the following are true:

- C_n is a monotone decreasing sequence
- C_n converges to 0
- $\sum_{n=0}^{\infty} C_n$ diverges
- $\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$ has radius of convergence 4.

What is the center of convergence of $\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

Answer: -3

What are the endpoints of the interval of convergence of $\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

Answer: Left endpoint at $a =$ -7

Right endpoint at $b =$ 1

Let a and b be the left and right endpoints of the interval of convergence you found above.
Which of the following could be the interval of convergence of $\sum_{n=0}^{\infty} \frac{C_n(x+3)^n}{4^n}$?

$[a, b]$ $\quad$ $(a, b]$ $\quad$ (a, b) $\quad$ $\boxed{[a, b)}$

8. [12 points] Let $p(x)$ be the probability density function given by

$$p(x) = \begin{cases} \dfrac{c}{x^2} & \text{for } x \le -1 \\ c & \text{for } -1 < x < 1 \\ \dfrac{c}{x^2} & \text{for } x \ge 1. \end{cases}$$

for some value c.

a. [5 points] Find the value of c. Justify your answer. Any integrals must be computed by hand.

Solution: We know that for any pdf $p(x)$, we must have $\int_{-\infty}^{\infty} p(x)\,dx = 1$. In this case, that means

$$\int_{-\infty}^{-1} c/x^2\,dx + \int_{-1}^{1} c\,dx + \int_{1}^{\infty} c/x^2\,dx = 1.$$

Changing to limits gives

$$\lim_{a\to-\infty} \int_{a}^{-1} c/x^2\,dx + \int_{-1}^{1} c\,dx + \lim_{b\to\infty} \int_{1}^{b} c/x^2\,dx = 1.$$

An antiderivative of c/x^2 is $-c/x$, so the line above can be written as

$$\lim_{a\to-\infty} (-c/(-1) - (-c/a)) + 2c + \lim_{b\to\infty} ((-c/b) - (-c/1)) = 1$$

Since $\lim_{a\to-\infty} -c/a = \lim_{b\to\infty} -c/b = 0$, we have $4c = 1$ or $c = 1/4$.

Answer: $c =$ **1/4**

b. [4 points] Find a piecewise-defined formula for $P(x)$, the cumulative density function for x.

Solution: The CDF is an antiderivative of the PDF. We need to choose correct constants to add on each piece so that $P(x)$ is increasing, $\lim_{x\to-\infty} P(x) = 0$, and $\lim_{x\to\infty} P(x) = 1$. This happens when

$$P(x) = \begin{cases} -\dfrac{c}{x} & \text{for } x \le -1 \\ cx + 2c & \text{for } -1 < x < 1 \\ 1 - \dfrac{c}{x} & \text{for } x \ge 1. \end{cases}$$

c. [3 points] Show that there is no mean value of x.

Solution: We find the mean by finding $\int_{-\infty}^{\infty} xp(x)\,dx$. In this case, that means

$$\int_{-\infty}^{-1} c/x\,dx + \int_{-1}^{1} cx\,dx + \int_{1}^{\infty} c/x\,dx.$$

If any of these diverge, then the integral from $-\infty$ to ∞ also diverges, and $\int_{1}^{\infty} c/x\,dx$ diverges by p-test, $p = 1$.

9. [9 points] The blueprint for the Infinity Tower has been finalized, and the design of the Tower of Hanoi is accepted. Specifically:

- the tower will have infinitely many floors
- each floor has the shape of a solid cylinder of height of 3 meters
- the nth floor has radius $\dfrac{1}{2n^2}$ meters
- the ground floor corresponds to $n = 1$
- the tower has constant density δ kg/m^3
- when construction begins, all materials are on the ground and have to be lifted to build each floor.

In this problem, you may assume the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [7 points] Let W_n be the work, in Joules, it takes to lift the materials to build the nth floor and put that floor in place in the tower. Write an expression involving one or more integrals for each of the following.

i. $W_1 = \int_0^3 \pi \left(\frac{1}{2}\right)^2 \delta g h \, dh$

ii. $W_2 = \int_3^6 \pi \left(\frac{1}{8}\right)^2 \delta g h \, dh = \int_0^3 \pi \left(\frac{1}{8}\right)^2 \delta g (3+h) \, dh$

iii. $W_n = \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h \, dh = \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1)+h) \, dh$

b. [2 points] Write an expression involving one or more integrals and/or series that gives the total work it would take to build the entire tower. Your answer should not include the letter W.

Answer: $\sum_{n=1}^{\infty} \int_{3(n-1)}^{3n} \pi \left(\frac{1}{2n^2}\right)^2 \delta g h \, dh = \sum_{n=1}^{\infty} \int_0^3 \pi \left(\frac{1}{2n^2}\right)^2 \delta g (3(n-1)+h) \, dh$

10. [9 points] Let $f(x)$ be a continuous, decreasing, positive function defined for all $0 < x < \infty$ such that

- $\int_0^1 f(x)\,dx$ diverges and
- $\int_1^\infty f(x)\,dx$ converges.

Define $a_n = f(n)$ and $S_n = a_1 + a_2 + \cdots + a_n$ for $n \geq 1$.

Determine whether each of the following must converge, must diverge, or whether convergence cannot be determined. You do not need to justify your answers.

i. The sequence a_n $\boxed{\textbf{Converges}}$ **Diverges** **Cannot be determined**

ii. The series $\sum_{n=1}^{\infty} a_n$ $\boxed{\textbf{Converges}}$ **Diverges** **Cannot be determined**

iii. The sequence S_n $\boxed{\textbf{Converges}}$ **Diverges** **Cannot be determined**

iv. The series $\sum_{n=1}^{\infty} S_n$ **Converges** $\boxed{\textbf{Diverges}}$ **Cannot be determined**

v. $\int_0^\infty f(x)\,dx$ **Converges** $\boxed{\textbf{Diverges}}$ **Cannot be determined**

vi. The series $\sum_{n=1}^{\infty} f(1/n)$ **Converges** $\boxed{\textbf{Diverges}}$ **Cannot be determined**

vii. $\int_0^1 \frac{f(1/x)}{x^2}\,dx$ $\boxed{\textbf{Converges}}$ **Diverges** **Cannot be determined**