UNOFFICIAL SOLUTIONS

On my honor, as a student, I have neither given nor received unauthorized aid on this academic work. **Initials:** ______

Do not write in this area

Math 116 — Final Exam — April 26, 2019

Your Initials Only: ______ Your U-M ID # (not uniqname): ______________________

Instructor Name: ______________________________________ Section #: ________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. The last page provides some potentially useful formulas. You may separate the formula page from the exam, but please do turn it in along with the exam. Otherwise, do not separate the pages of this exam. If pages do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	8	
3	10	
4	8	
5	9	

Problem	Points	Score
6	11	
7	13	
8	12	
9	7	
10	8	
Total	100	

1. [14 points] Hannah Haire and Ryan Rabbit meet for one last race. Once again, they both start at the west side of a large square field that is 10 km wide; it will end when one reaches the east side. The racers' (x, y) positions are given by the parametric equations below, where $(0, 0)$ represents the southwest corner of the field, x represents kilometers east of this corner, y represents kilometers north of this corner, and $t \geq 0$ is measured in hours after the race begins.

Hannah Haire: $\begin{cases} x = t^2 \\ y = \dfrac{t^2}{2} + 2 \end{cases}$ $x'(t) = 2t$, $y'(t) = t$

Ryan Rabbitt: $\begin{cases} x = 4t - t^2 \\ y = t^2 - t + 1 \end{cases}$ $x'(t) = 4 - 2t$, $y'(t) = 2t - 1$

Be sure to justify your answers to the following questions algebraically.

a. [2 points] Who is going faster two hours into the race?

Hannah: Speed $= \sqrt{x'(2)^2 + y'(2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$

Ryan: Speed $= \sqrt{x'(2)^2 + y'(2)^2} = \sqrt{0^2 + 3^2} = 3$

Answer: Hannah

b. [3 points] The race ends when the first racer reaches the east side of the field. When does the race end? Who wins?

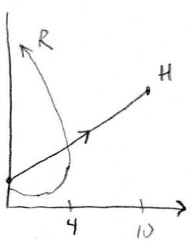
Hannah finishes when: $t^2 = 10 \Rightarrow t = \sqrt{10}$

Ryan finishes when: $4t - t^2 = 10 \Rightarrow t^2 - 4t + 10 = 0 \Rightarrow t = \dfrac{4 \pm \sqrt{16 - 40}}{2}$

No real solutions, so Ryan never finishes.

Answer: Race ends at $t = \sqrt{10}$ **Winner:** $\boxed{\text{Hannah}}$ Ryan Tie

c. [3 points] Write an integral representing the distance, in km, that Ryan runs during the race.



$\int_0^{\sqrt{10}} (\text{Ryan's speed})\, dt$

Answer: $\int_0^{\sqrt{10}} \sqrt{(4-2t)^2 + (2t-1)^2}\, dt$

d. [3 points] Find all times at which Ryan and Hannah are in the same spot on the field. If there are none, write "none".

x-values match when $t^2 = 4t - t^2 \Rightarrow 0 = 4t - 2t^2 = 2t(2-t)$
$\Rightarrow t = 0$ or $t = 2$.

But y-values match at neither $t = 0$ nor $t = 2$.

Answer: $t =$ NONE

e. [3 points] Find all times at which Ryan is facing directly northeast (that is, halfway between directly north and directly east). If there are none, write "none".

NE $\Rightarrow \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dt} = \dfrac{dx}{dt} > 0$.

So $4 - 2t = 2t - 1 \Rightarrow 5 = 4t \Rightarrow t = 1.25$

At that time,

$\dfrac{dx}{dt} = 4 - 2.5 = 1.5$, $\dfrac{dy}{dt} = 2.5 - 1 = 1.5$, so both are positive.

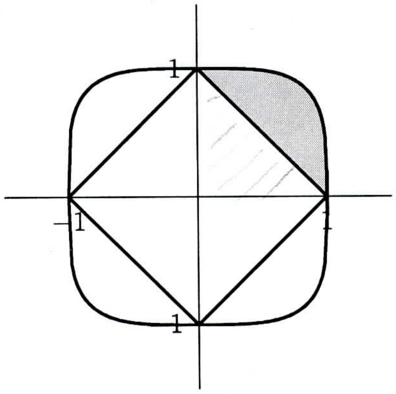
Answer: $t =$ 1.25

2. [8 points]

The *castar*, a coin widely used in Middle-Earth, allegedly has the shape graphed to the right. The outer perimeter can be modeled by the implicit equation

$$x^4 + y^4 = 1$$

and the perimeter of the hole in the middle is a square. To help his fellow Hobbits detect counterfeit coins, Samwise Gamgee, the Mayor of the Shire, is working on obtaining the specifications of a genuine castar. Sam needs your help.



a. [2 points] Find a function $f(\theta)$ so that the outer edge of the castar is given by the function $r = f(\theta)$.

$x = r\cos\theta$, $y = r\sin\theta$, so

$(r\cos\theta)^4 + (r\sin\theta)^4 = 1$

$\Rightarrow r^4\cos^4\theta + r^4\sin^4\theta = 1$

Answer: $f(\theta) = \left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/4}$

b. [3 points] Write an expression involving one or more integrals that gives the total area of the quarter of a castar in the first quadrant (shaded above).

Area of shaded triangle $= \frac{1}{2}$, so

area of quarter coin $= \frac{1}{2}\int_0^{\pi/2} f(\theta)^2\,d\theta - \frac{1}{2}$

Answer: $\dfrac{1}{2}\displaystyle\int_0^{\pi/2}\left(\dfrac{1}{\cos^4\theta + \sin^4\theta}\right)^{1/2} d\theta - \dfrac{1}{2}$

c. [3 points] Approximate the area of a castar by estimating your integral(s) from part (b) using TRAP(2). Write out all the terms in your sum(s).

For $\int_0^B g(x)\,dx$, $\text{TRAP}(2) = \Delta x\left[\frac{1}{2}g(0) + g\left(\frac{B}{2}\right) + \frac{1}{2}g(B)\right]$, where $\Delta x = \frac{B}{2}$.

In our case $B = \frac{\pi}{2}$ so $\frac{B}{2} = \Delta x = \frac{\pi}{4}$.

four quarters

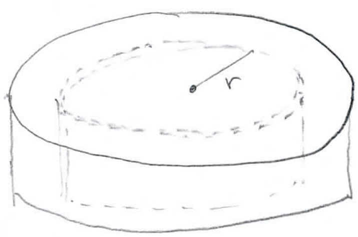
Answer: $4\left[\frac{1}{2}\cdot\frac{\pi}{4}\left[\frac{1}{2}\left(\frac{1}{\cos^4(0)+\sin^4(0)}\right)^{1/2} + \left(\frac{1}{\cos^4(\frac{\pi}{4})+\sin^4(\frac{\pi}{4})}\right)^{1/2} + \frac{1}{2}\left(\frac{1}{\cos^4(\frac{\pi}{2})+\sin^4(\frac{\pi}{2})}\right)^{1/2}\right] - 1/2\right]$

$= 4\left[\frac{\pi}{8}(1+\sqrt{2}) - \frac{1}{2}\right] = \frac{\pi}{2}(1+\sqrt{2}) - 2 \approx 1.79$

3. [10 points] A group of scientists of S.H.I.E.L.D. are investigating the Battle of Sokovia, trying to understand how Ultron lifted the capital city of Sokovia up into the sky. They use data available to them to model the situation.
Pay careful attention to the units involved in the data they use.

a. [5 points] The scientists find that they can model the part of the city that was lifted by the shape of a cylinder of radius 2 kilometers and height 100 meters. The density $\delta(r)$, in kilograms per cubic meter, is a function of distance r meters away from the central axis of the cylinder. Let M be the total mass, in kilograms, of the part of the city that was lifted. Write an expression involving one or more integrals that gives the value of M.

Density in kg/m³, so convert km to m



Shape of slice = cylindrical shell

radius " " = r $(0 \le r \le 2000\text{ m})$

height " " = 100 m

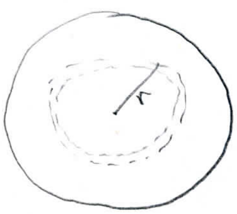
area " " = $(2\pi r)(\text{height}) = 200\pi r\ \text{m}^2$

thickness " " = Δr m

Volume " " = $200\pi r\,\Delta r\ \text{m}^3$

mass " " = (volume)(density) = $200\pi r\,\delta(r)\Delta r$ kg

Top view



Answer: $M =$ $\int_0^{2000} 200\pi r\,\delta(r)\,dr$ kg

b. [5 points] You may use M and $\delta(r)$ from part **a.** for this part.
Ultron lifted the city at a constant rate of 2 meters per second to a height of 1000 meters above the ground. While he lifted it, a small portion of the city kept detaching from the rising part at a constant rate of p kilograms per second. Write an expression involving one or more integrals that gives the total work, in Joules, it takes to complete the lifting process. Your answer may be in terms of $m, g, \delta(r)$, and M, where g is the gravitational constant, $g \approx 9.8\text{ m/s}^2$.

Total Time to lift = $1000\text{ m}/2\text{ m/s} = 500$ s

mass at time t = $M - pt$ kg

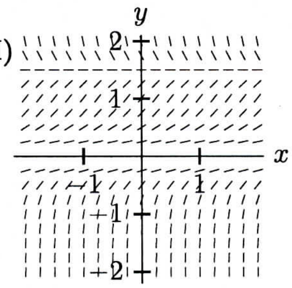
weight " " " = $(M-pt)g$ N

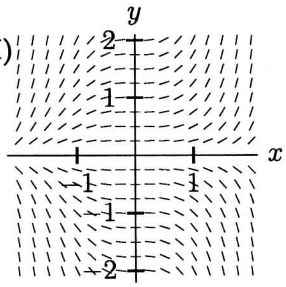
Distance lifted from time t to $t+\Delta t$ = $2\Delta t$ m

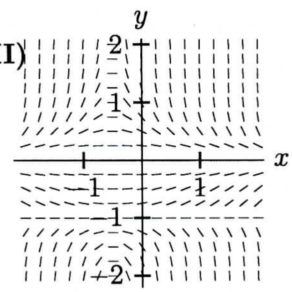
work done from time t to $t+\Delta t$ = (weight)(distance) = $2(M-pt)g\,\Delta t$ J

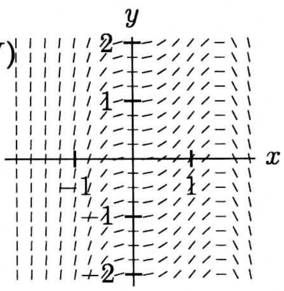
Answer: $\int_0^{500} 2(M-pt)g\,dt$ joules

4. [8 points] Four slope fields are given below.

(I)


(II)


(III)


(IV)


$ax^2(x-b)$ doesn't depend on y, so slopes should not vary as you travel along a vertical line. That eliminates I, II, and III.

a. [4 points] Suppose a and b are constants. Which one of the slope fields above could be the slope field for the differential equation $\frac{dy}{dx} = ax^2(x-b)$? (Circle one.)

(I) **(II)**

(III) $\boxed{\textbf{(IV)}}$

Sign of slope changes at approx $x = 1.5$, so that's b.

Based on this slope field, which of the following **must** be true about a?

$a > 0$ $a = 0$ $\boxed{a < 0}$ NONE OF THESE

For $x < 0$, $\frac{dy}{dx} > 0$, $x^2 > 0$, $x - b < 0$ so a must be < 0.

Based on this slope field, which of the following **must** be true about b?

$\boxed{b > 0}$ $b = 0$ $b < 0$ NONE OF THESE

b. [4 points] Find all equilibrium solutions of slope field **(I)** (on the upper left side) and determine whether they are stable. If there are no equilibrium solutions, write "none".

Stable equilibrium solutions:	**Other equilibrium solutions:**
$y = 1.5$	$y = 0$

5. [9 points] Determine whether each of the following series converges or diverges. Fully justify your answer, including carefully showing all work for any computations. Include any convergence tests used.

a. [4 points] $\displaystyle\sum_{n=1}^{\infty} \frac{3-\sin(n^4)}{n^2}$

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Justification:

$-1 \leq \sin(n^4) \leq 1$

So $2 \leq 3-\sin(n^4) \leq 4$

So $\dfrac{2}{n^2} \leq \dfrac{3-\sin(n^4)}{n^2} \leq \dfrac{4}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges by the p-test $(p=2)$

So since $\dfrac{3-\sin(n^4)}{n^2}$ is positive, $\displaystyle\sum \frac{3-\sin(n^4)}{n^2}$ converges by comparison.

b. [5 points] $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Justification:

Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x\sqrt{\ln x}}\,dx$$

let $w = \ln x$

$dw = \frac{1}{x}\,dx$

$x=2 \Rightarrow w = \ln 2$

$x=b \Rightarrow w = \ln b$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{\sqrt{w}}\,dw = \int_{\ln 2}^{\infty} \frac{1}{w^{1/2}}\,dw$$

which diverges by the p-test $(p=\frac{1}{2})$.

So since $\dfrac{1}{n\sqrt{\ln n}}$ is positive and decreasing,

$\displaystyle\sum_{2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges by the integral test.

6. [11 points] Consider the function $g(x)$ defined for all real numbers represented by the Taylor series

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n-1}}{(n+1)!} x^{2n}.$$

a. [3 points] Find the values of $g^{(2019)}(0)$ and $g^{(2020)}(0)$. You do *not* need to simplify.

Answer: $g^{(2019)}(0) =$ 0 $\qquad g^{(2020)}(0) =$ $\boxed{-2^{2019} \frac{2020!}{1011!}}$

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k$$

. Since all terms in the sum have even degree, $g^{(2019)}(0) = 0$. There is an x^{2020} term, when $n = 1010$.

So $\frac{g^{(2020)}(0)}{2020!} = (-1)^{1010-1} \frac{2^{2020-1}}{(1010+1)!}$

b. [2 points] Find $P_4(x)$, the Taylor polynomial of $g(x)$ of degree 4 near $x = 0$.

$n=1$: $(-1)^{1-1} \frac{2^{2\cdot 1-1}}{(1+1)!} x^{2\cdot 1} = x^2$

$n=2$: $(-1)^{2-1} \frac{2^{2\cdot 2-1}}{(2+1)!} x^{2\cdot 2} = -\frac{8}{6} x^4$

Answer: $P_4(x) =$ $x^2 - \frac{4}{3}x^4$

c. [3 points] Define

$$G(x) = \int_{-1}^{x} g(t)\, dt.$$

Use $P_4(x)$ to estimate $G(2)$.

Answer: $G(2) \approx$ -5.8

$$G(2) = \int_{-1}^{2} g(t)\,dt \approx \int_{-1}^{2} P_4(t)\,dt = \int_{-1}^{2} t^2 - \frac{4}{3}t^4\,dt = \frac{1}{3}t^3 - \frac{4}{15}t^5 \Big|_{-1}^{2}$$

$$= \left[\frac{1}{3}(2)^3 - \frac{4}{15}(2)^5\right] - \left[\frac{1}{3}(-1)^3 - \frac{4}{15}(-1)^5\right] = \frac{8}{3} - \frac{128}{15} + \frac{1}{3} - \frac{4}{15}$$

d. [3 points] Use an appropriate Taylor polynomial to compute the limit

$$\lim_{x \to 0+} \frac{g'(x)}{x}$$

Show your work carefully.

Answer: 2

$g(x) = x^2 +$ some terms of degree at least 4

so $g'(x) = 2x +$ " " " " " 3

So $\frac{g'(x)}{x} = 2 + \underbrace{\text{" " " " " } 2}_{\text{these} \to 0 \text{ as } x \to 0}$

7. [13 points] The parts of this problem are unrelated.

a. [3 points] Consider the function

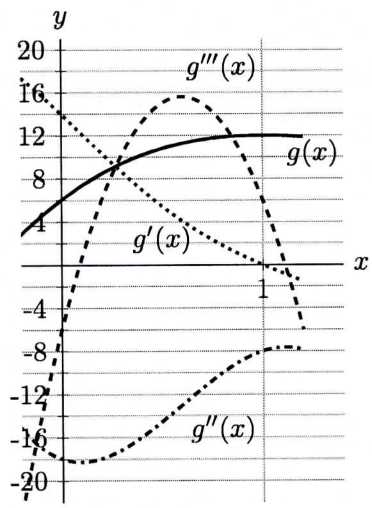
$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{\sin x}{x} - \cos x & \text{for } x \neq 0 \end{cases}$$

Find the Taylor series for $f(x)$ centered at $x = 0$. Write your answer as a single sum using sigma notation.

$f(x) = \frac{1}{x}\sin(x) - \cos(x) = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n+1)!} - \frac{1}{(2n)!}\right]x^{2n}$

Answer: $f(x) =$ $\sum_{n=0}^{\infty}(-1)^n\left[\frac{-2n}{(2n+1)!}\right]x^{2n}$

b. [4 points] Part of the graphs of $g(x), g'(x), g''(x)$, and $g'''(x)$ are given to the right. Find the third-degree Taylor polynomial for $g(x)$ near $x = 1$.



$g(1) = 12$
$g'(1) = 0$
$g''(1) = -8$
$g'''(1) = 6$

$P_3(x) = g(1) + g'(1)(x-1) + \frac{1}{2}g''(1)(x-1)^2 + \frac{1}{6}g'''(1)(x-1)^3$

$= 12 + (0)(x-1) + \frac{1}{2}(-8)(x-1)^2 + \frac{1}{6}(6)(x-1)^3$

Answer: $12 - 4(x-1)^2 + (x-1)^3$

c. [6 points] Find the exact value (in closed form) of the following series. You do not need to justify your answers.

i. $0.1 + \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.0001}{4} + \cdots =$ $-\ln(.9) = \ln 10 - \ln 9$

ii. $\frac{\pi}{2} - \frac{3}{\pi} + \frac{18}{\pi^3} - \frac{108}{\pi^5} + \cdots =$ $\frac{\pi}{2}\left[1 - \frac{6}{\pi^2} + \frac{6^2}{\pi^4} - \frac{6^3}{\pi^6} + \cdots\right]$ $\pi^3/(2\pi^2+12)$

iii. $\frac{1}{2} - 2e^2 + \frac{2^3e^4}{3!} - \frac{2^5e^6}{5!} + \cdots =$ $\frac{1}{2} - e\left[2e - \frac{2^3e^3}{3!} + \frac{2^5e^5}{5!} - \cdots\right]$ $\frac{1}{2} - e\sin(2e)$

i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

So $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

So $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

So $-\ln(1-.1) = .1 + \frac{.01}{2} + \frac{.001}{3} + \frac{.0001}{4} + \cdots$

ii) is geometric with $a = \frac{\pi}{2}$, $x = \frac{-6}{\pi^2}$

So $\frac{\pi/2}{1 - \frac{-6}{\pi^2}} \cdot \frac{2\pi^2}{2\pi^2} = \frac{\pi^3}{2\pi^2+12}$

iii) Looks like sin:

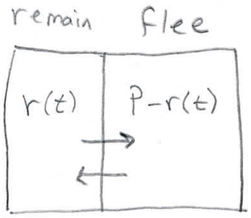
$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

8. [12 points] The Resistance suddenly find themselves in a huge crisis! Chased by First Order's fleet, the Resistance members are deciding whether to evacuate their starship and flee in small transports, or to remain and fight. At every moment throughout the debate, every Resistance member is voting either to remain or flee, but members are continuously changing their vote. Votes change in the following way:

- The number of Resistance members who change their vote from fleeing to remaining is proportional to the number that is currently voting to flee
- The number of Resistance members who change their vote from remaining to fleeing is proportional to the number that is currently voting to remain
- These both have the **same** constant of proportionality k, where $k > 0$.

a. [4 points] Let P be the total number of Resistance members, and let $r(t)$ be the number of members vote to remain t minutes after the debate begins. Write a differential equation for $r(t)$ which models the scenario.

Answer: $\dfrac{dr}{dt} =$ $k(P-r(t)) - kr(t)$



$= kP - 2kr(t)$

$= k(P - 2r(t))$

b. [3 points] Find all equilibrium solutions to your differential equation and determine whether they are stable. Interpret your answer in the context of the problem.

Equilibrium @ $\boxed{r = \frac{P}{2}}$ Stable solution when the vote is a tie!

If $r > \frac{P}{2}$, $P - 2r < 0$, so $\frac{dr}{dt} < 0$ dec

if $r < \frac{P}{2}$, $P - 2r > 0$, so $\frac{dr}{dt} > 0$ inc

$\Rightarrow$ Stable

c. [5 points] At a moment when 60% of the Resistance members wish to remain, Princess Leia recovers, and the situation drastically changes. Now let R be the fraction of Resistance members who wish to remain t seconds after Leia's recovery, and suppose R satisfies the differential equation

$$\frac{dR}{dt} = -Re^t.$$

Find an explicit formula for $R(t)$. Show your work carefully.

$\int \frac{dR}{R} = \int -e^t\,dt \Rightarrow \ln R = -e^t + C \Rightarrow R = Ae^{-e^t}$

$.6 = R(0) = Ae^{-e^0} = Ae^{-1} \Rightarrow A = .6e$

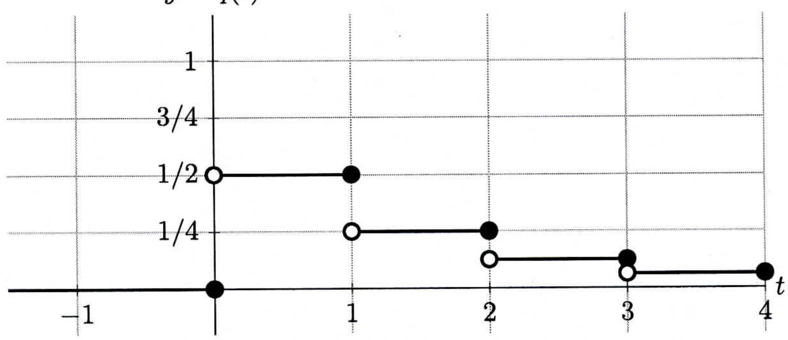
Answer: $R(t) =$ $.6e \cdot e^{-e^t} = \boxed{.6e^{1-e^t}}$

9. [7 points] After the first ever picture of a black hole was released by Event Horizon Telescope (EHT), the public awaits an image of Sgr A*, the black hole at the Galactic Center. Let t be the amount of time, in years, between now and when the EHT releases such an image. The probability density function for t is given by

$$q(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1/2^n & \text{for } \cancel{n \le t < n+1}\ n-1 < t \le n, \text{ for each positive integer } n. \end{cases}$$

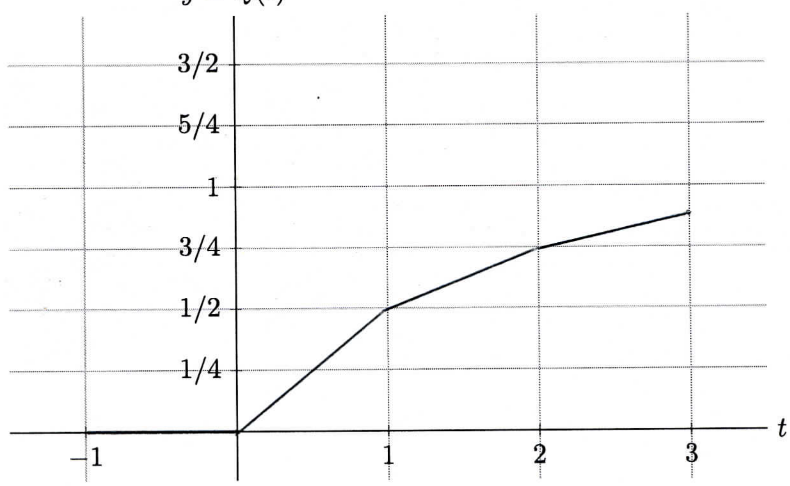
Part of the graph of $q(t)$ is given below.

$y = q(t)$



a. [4 points] Let $Q(t)$ be the cumulative distribution function for t. Carefully sketch the graph of $Q(t)$ on the domain $-1 \le t \le 3$.

$y = Q(t)$



b. [3 points] Let P_n be the probability that EHT releases an image of Sgr A* within n years, and p_n be the probability that release time is in the nth year.
For each part below, circle "True" if the statement **must be** true and circle "False" otherwise. No justification is necessary.

$p_n = \int_{n-1}^{n} q(t)\,dt = \frac{1}{2^n}$

The sequence p_n converges. $\frac{1}{2^n} \to 0$ $\boxed{\text{TRUE}}$ FALSE

The sequence P_n converges to 0. $\sum_{1}^{\infty} P_n = \frac{1/2}{1-1/2} = 1$ $\cancel{\boxed{\text{TRUE}}}$ $\boxed{\text{FALSE}}$

The series $\sum_{n=1}^{\infty} p_n$ converges. // $\boxed{\text{TRUE}}$ FALSE

(converges to 1)

10. [8 points] The following problems are unrelated.

a. [3 points] Which of the following are solutions to the differential equation $y' = x + y$? Circle all correct answers.

$\boxed{\text{i. } y = -1 - x + 3e^x}$ $y' = -1 + 3e^x$

ii. $y = 1 - x + 9e^x$ $y' = -1 + 9e^x$

iii. $y = -2 - x + e^x$ $y' = -1 + e^x$

$\boxed{\text{iv. } y = -1 - x + 7e^x}$ $y' = -1 + 7e^x$

v. $y = e^x + x$ $y' = e^x + 1$

vi. $y = e^{x^2/2}$ $y' = xe^{x^2/2}$

vii. NONE OF THESE

$F(x)$
$\parallel$

b. [3 points] Suppose $\sum_{n=0}^{\infty} a_n(x-2)^n$ is a power series with interval of convergence $(-1, 5]$. Which of the following statements **must** be true? Circle all that are correct.

$F(5)$
$\parallel$
$\boxed{\text{i. } \sum_{n=0}^{\infty} 3^n a_n \text{ converges conditionally.}}$

ii. $\sum_{n=0}^{\infty} 3^n a_n$ converges absolutely.

iii. $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 3$

$f(3)$
$\parallel$
iv. $\sum_{n=0}^{\infty} a_n$ converges conditionally.

$\boxed{\text{v. } \sum_{n=0}^{\infty} a_n \text{ converges absolutely.}}$

vi. $\sum_{n=1}^{\infty} \frac{|a_n|}{n}$ diverges.

vii. NONE OF THESE

$|3^n a_n|$
$= |(-3)^n a_n|$
$\geq (-3)^n a_n$
$\sum (-3)^n a_n$
diverges since it's what you get when you plug in -1.
So $\sum |3^n a_n|$ diverges, $\Rightarrow \sum 3^n a_n$ converges conditionally

c. [2 points] For what value of β does $\int_{\pi/18}^{\beta} \sqrt{\sin^2(3\theta) + 9\cos^2(3\theta)}\, d\theta$ give the length of the arc along the polar curve $r = \sin(3\theta)$ in the first quadrant and outside the circle $r = 1/2$? Circle the **one** best answer.

i. $-\pi/18$

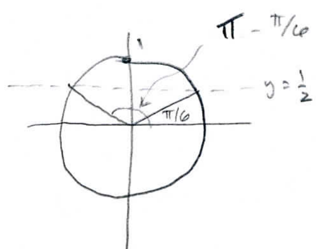
ii. $\pi/18 + 2\pi$

iii. $\pi - \pi/18$

iv. $\pi/2 - \pi/18$

$\boxed{\text{v. } \pi/3 - \pi/18}$

vi. NONE OF THESE



$\sin 3\theta \geq \frac{1}{2}$ if $\frac{\pi}{6} \leq 3\theta \leq \pi - \frac{\pi}{6}$

$\Longleftrightarrow \frac{\pi}{18} \leq \theta \leq \frac{\pi}{3} - \frac{\pi}{18}$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Normal Distributions

The density function of a normal distribution with mean μ and standard deviation $\sigma > 0$ is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

The standard normal distribution is the normal distribution with $\mu = 0$ and $\sigma = 1$.