

Math 116 — Second Midterm — March 29, 2021

1. You must write your solutions, including all work, for this exam on blank paper and upload your solutions to Gradescope when the exam is over. You may not write your answers on a printed copy of this exam.
 2. Write your UMID on the upper right corner of every page you submit. Do not write your name on your submission.
 3. This exam has 9 pages including this cover.
 4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
 5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
 6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
 7. The use of any digital or computational device while working on this exam is not permitted. This means you may NOT use a calculator, tablet, computer or other device to do the problems on the exam.
 8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
 9. Include units in your answer where that is appropriate.
 10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
 11. You may use any physical notes or books. These may be handwritten or typed.
 12. You must use the methods learned in this course to solve all problems.
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Problem	Points	Score
1	6	
2	12	
3	7	
4	7	
5	12	

Problem	Points	Score
6	12	
7	6	
8	9	
9	17	
10	12	
Total	100	

1. [6 points] This question is about exam policies. These are the same questions sent to you before the exam. The answer to each is either Yes or No. For each question, write the letter corresponding to the question and the entire word YES or the entire word NO. You do not need to rewrite the questions on your paper.
- a. Are you allowed to use any resources, typed or handwritten, by accessing them on a digital device (e.g. computer, tablet, or phone)?
 - b. Are you allowed to refer to a physical copy of the textbook during the exam?
 - c. Are you allowed to use a calculator on the exam?
 - d. If you notice a mistake in one of your answers while you are scanning, is it okay to correct that mistake?
 - e. Can the penalty for cheating be a failing grade in the course?
 - f. Is 0.33333333 an exact answer to the equation $3x = 1$?

2. [12 points] In order to build a settlement on the island, intruders start cutting down trees at the forest, cutting the trees into logs, and putting the logs in a pile. Let A_n be the number of logs they have in the pile at noon on the n -th day. The intruders have 100 logs in the pile at noon on the first day (so $A_1 = 100$). Every day (between noon on one day and noon on the next day), the building team uses 10% of the logs in the pile, while the log-cutting team adds 20 logs to the pile immediately before noon.

a. [4 points] Find A_2 and A_3 . **You do not need to simplify your answers.**

b. [5 points] Find a **closed form** expression for A_n . Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. **You do not need to simplify your closed form answer.**

c. [3 points] How many logs will the intruders have in the pile in the long run?

3. [7 points] Emily is a physics teacher. She is demonstrating the physics of rocket-launching with a water rocket in her class today. The water rocket weighs 3 lbs on the ground, and Emily launches it straight up to 10 ft above the ground. During the launch, the rocket's weight decreases at constant rate (in lbs/ft) as the water is ejected from the rocket. When it reaches 10 ft above the ground, the rocket weighs 1 lb.
- a. [3 points] Calculate the weight of the rocket when it is a height h ft above the ground. Include units.
- b. [4 points] Write an expression involving integrals for the total work required to propel the rocket from the ground to a height of 10 feet above the ground (as described above). Do not evaluate any integrals in your expression. Include units.

4. [7 points] Some of Flora's friends are hurt while the intruders are building the settlement. Flora and Nile are trying their best to heal them. Suppose $p(x)$ is the probability density function for the number of weeks, x , it takes for everyone to recover after intruders appear.

$$p(x) = \begin{cases} c & \text{if } 0 < x \leq 1, \\ 2c & \text{if } 1 < x \leq 3, \\ 0 & \text{else.} \end{cases}$$

- a. [2 points] Find c .
- b. [5 points] Let $W(x)$ be the cumulative distribution function for $p(x)$. Showing your work, give a piece-wise defined formula for $W(x)$ in the form given below.

$$W(x) = \begin{cases} \underline{\hspace{2cm}} & \text{if } x \leq 0, \\ \underline{\hspace{2cm}} & \text{if } 0 < x \leq 1, \\ \underline{\hspace{2cm}} & \text{if } 1 < x \leq 3, \\ \underline{\hspace{2cm}} & \text{if } x > 3. \end{cases}$$

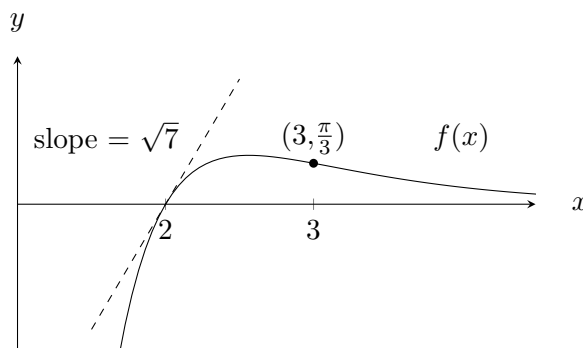
5. [12 points] Another function $f(t)$ given by

$$f(t) = \begin{cases} \frac{t}{6} & \text{if } 0 < t \leq 2, \\ \frac{1}{3} & \text{if } 2 < t \leq 4, \\ 0 & \text{else.} \end{cases}$$

is the probability density function for the number of months t that it will take the intruders to build the settlement.

- a. [3 points] Find the probability that it will take the intruders between 1 and 2 months to build the settlement.
- b. [5 points] Find the mean number of months it will take the intruders to build the settlement.
- c. [4 points] Find the median number of months it will take the intruders to build the settlement.

6. [12 points] Below is the graph of a function $f(x)$. The function $f(x)$ is positive for $x > 2$. The x -axis is a horizontal asymptote of $f(x)$ as $x \rightarrow \infty$. The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is $\sqrt{7}$.



a. [2 points] Compute $\lim_{x \rightarrow 2^+} \tan^{-1} \frac{1}{f(x)}$.

(*Hint:* Recall that $\lim_{z \rightarrow \infty} \tan^{-1} z = \frac{\pi}{2}$ and $\lim_{z \rightarrow -\infty} \tan^{-1} z = -\frac{\pi}{2}$.)

b. [3 points] Compute $\lim_{x \rightarrow 2^-} \frac{f(x)}{e^x - e^2}$.

- c. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation**.

$$\int_3^{\infty} \frac{f'(x)}{(f(x))^{2/3}} dx$$

7. [6 points] Determine whether the following improper integral converges or diverges. **Fully justify** your answer including using **proper notation**, and showing mechanics of any tests or theorems you use.

$$\int_0^1 \frac{\pi}{x^3 + \sqrt{x}} dx$$

8. [9 points] Consider the following 4 sequences.

$$(A) \quad a_n = (-1)^n,$$

$$(B) \quad b_n = 3 \cdot (0.5)^n,$$

$$(C) \quad c_n = \sum_{k=1}^n \frac{1}{k},$$

$$(D) \quad d_n = \int_0^n \frac{x}{e^x} dx$$

For each of the following, write down the CAPITAL LETTER corresponding to each of the sequences that satisfy the given property. **No justification is required.**

a. [3 points] Which sequence(s) is/are bounded?

b. [3 points] Which sequence(s) is/are monotone?

c. [3 points] Which sequence(s) is/are convergent?

9. [17 points]

- a. [4 points] Ivan is studying the series $\sum_{n=1}^{\infty} (-1)^n (1 + e^{-n})$ and writes the following argument:

The series is alternating. If we let $a_n = |(-1)^n (1 + e^{-n})| = 1 + e^{-n}$, then a_n is positive, decreasing, but $\lim_{n \rightarrow \infty} a_n = 1$ is not 0. Therefore $\sum_{n=1}^{\infty} (-1)^n (1 + e^{-n})$ diverges by the alternating series test.

Ivan's instructor tells Ivan that even though the $\sum_{n=1}^{\infty} (-1)^n (1 + e^{-n})$ does diverge, the above argument is incorrect. Explain what's wrong with this argument, and give a correct argument to show that $\sum_{n=1}^{\infty} (-1)^n (1 + e^{-n})$ diverges.

- b. [6 points] Determine whether the following series converge or diverge. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$$

- c. [7 points] Determine whether the following series **converges absolutely, converges conditionally, or diverges**. Be sure to fully justify your answer, using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

10. [12 points] Show that the following statements are false by giving a concrete example to contradict each of the statement. You can write a formula or draw a clear, well-labeled graph in place of the blanks. Accompany your example with a brief but complete explanation.

a. [4 points] If $\lim_{n \rightarrow \infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Give your answer in the form:

“The statement is false when $a_n =$ _____ because...”

b. [4 points] For any continuous function $f(x)$ with $f(x) > 0$, the improper integral $\int_{-100}^{\infty} f(x) dx$ always diverges.

Give your answer in the form:

“The statement is false when $f(x) =$ _____ because...”

c. [4 points] If $P(x)$ is a cumulative distribution function, then $P(0) = 0$.

Give your answer in the form:

“The statement is false when $P(x) =$ _____ because...”

(Note: Your $P(x)$ needs to be a cumulative distribution function, but you do not need to show/prove that it is.)