Math 116 — First Midterm — February 22, 2021

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1. You must write your solutions, including all work, for this exam on blank paper and upload you solutions to Gradscope when the exam is over. You may not write your answers on a printed copy of this exam.
2. Write your UMID on the upper right corner of every page you submit. Do not write your name on your submission.
3. This exam has 15 pages including this cover.
4. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any digital or computational device while working on this exam is not permitted. This means you may NOT use a calculator, tablet, computer or other device to do the problems on the exam.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You may use any physical notes or books. These may be handwritten or typed.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	13	
3	15	
4	10	
5	10	

Problem	Points	Score
6	15	
7	6	
8	13	
9	12	
Total	100	

1. [6 points] This question is about exam policies. These are the same questions sent to you before the exam. The answer to each is either Yes or No. For each question, write the letter corresponding to the question and the entire word YES or the entire word NO. You do not need to rewrite the questions on your paper.

a. Are you allowed to use any resources, typed or handwritten, by accessing them on a digital device (e.g. computer, tablet, or phone)?

b. Are you allowed to refer to a physical copy of the textbook during the exam?

c. Are you allowed to use a calculator on the exam?

d. If you notice a mistake in one of your answers while you are scanning, is it okay to correct that mistake?

e. Can the penalty for cheating be a failing grade in the course?

f. Is 0.333333333 an exact answer to the equation $3x = 1$?

Solution:

a. No

b. Yes

c. No

d. No

e. Yes

f. No

2. [13 points] Let $f(x) = \dfrac{1}{2x^2+1}$.

a. [4 points] Approximate the integral $\int_1^5 f(x)\, dx$ using MID(2). Write out each term in your sum. You do not need to simplify the numbers in your sum, but your final answer should not contain the letter "f".

Solution:

$$\text{MID}(2) = 2 \cdot \left(\frac{1}{2(2)^2+1} + \frac{1}{2(4)^2+1} \right)$$

b. [4 points] Approximate the integral $\int_1^5 f(x)\, dx$ using TRAP(2). You do not need to simplify the numbers in your sum, but your final answer should not contain the letter "f".

Solution:

$$\text{TRAP}(2) = 2 \cdot \frac{1}{2} \left(\frac{1}{2(1)^2+1} + \frac{2}{2(3)^2+1} + \frac{1}{2(5)^2+1} \right)$$

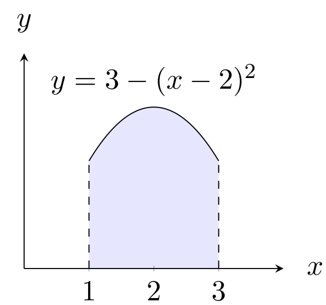
c. [5 points] Compute the exact value for $\int_1^5 f(x)\, dx$. Show all your steps. You do not need to simplify the numbers in your final answer.

Solution: Let $x = \frac{1}{\sqrt{2}} \tan\theta$. Then

$$\begin{aligned}
\int_1^5 \frac{1}{2x^2+1}\, dx &= \int_{\tan^{-1}\sqrt{2}}^{\tan^{-1}(5\sqrt{2})} \frac{1}{1+\tan^2\theta} \frac{1}{\sqrt{2}} \sec^2\theta \, d\theta \\
&= \frac{1}{\sqrt{2}} \int_{\tan^{-1}\sqrt{2}}^{\tan^{-1}(5\sqrt{2})} d\theta \\
&= \frac{1}{\sqrt{2}} (\tan^{-1}(5\sqrt{2}) - \tan^{-1}\sqrt{2}).
\end{aligned}$$

3. [15 points] Flora and Nile are collecting fruits in the forest and they have brought several containers of different shapes. **Write an integral** that computes the volume of each of the following containers. **Do not evaluate your integrals.**

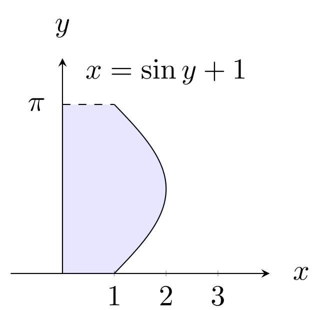
a. [5 points] The first container is in the shape formed by revolving the following region about the y-axis.



Solution:

$$\int_1^3 2\pi x(3-(x-2)^2)\,dx \text{ or } \pi(3^2-1^2)\cdot 2+\int_2^3 \pi((2+\sqrt{3-y})^2-(2-\sqrt{3-y})^2)\,dy$$

b. [5 points] The second container is in the shape formed by revolving the following region about the line $x = 3$.



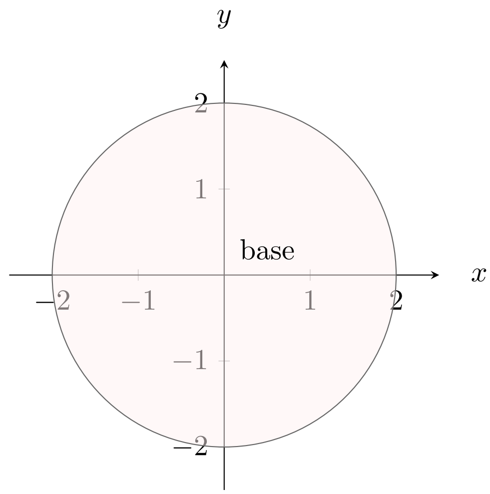
Solution:

$$\int_0^\pi \pi(3^2-(3-(\sin y+1))^2)\,dy$$

or

$$\pi(3^2-2^2)\cdot\pi+\int_1^2 2\pi(3-x)((\pi-\sin^{-1}(x-1))-\sin^{-1}(x-1))\,dx$$

c. [5 points] The third container has a circular base with equation $x^2 + y^2 = 4$ (of radius 2 centered at the origin), with **square** cross-sections perpendicular to the x-axis.



Solution:

$$\int_{-2}^{2} (2\sqrt{4-x^2})^2 \, dx$$

4. [10 points] After walking in the woods, Flora is making juice with the fruit she picked up at the next hour. The volume of juice in the jar (in gallons) t minutes after she starts making juice is given by the function

$$F(t) = \int_{\sin t}^{2t} \frac{50}{100 - \ln(x+2)}\, dx.$$

a. [3 points] Calculate $F'(t)$.

Solution:

$$F'(t) = \frac{50}{100 - \ln(2t+2)} \cdot 2 - \frac{50}{100 - \ln(\sin t + 2)} \cdot \cos t.$$

b. [3 points] What is the volume of juice (in gallons) in the jar when Flora starts making the juice? Briefly explain your answer using the function $F(t)$.

Solution:

$$F(0) = \int_0^0 (...) dx = 0.$$

c. [4 points] Nile wants to know the volume of juice in the jar, yet she is confused by the function $F(t)$. She knows she can write $F(t)$ using $F'(t)$ and the initial volume of juice in the jar. Help her by rewriting $F(t)$ in the form

$$F(t) = \int_a^t \underline{\qquad}\ d\underline{\qquad} + \underline{\qquad}.$$

Write the above integral with the blanks filled in, and also give the value of a.

Solution:

$$F(t) = \int_0^t \frac{50}{100 - \ln(2x+2)} \cdot 2 - \frac{50}{100 - \ln(\sin x + 2)} \cdot \cos x dx + 0.$$

5. [10 points] Flora pours herself a cup of juice in a cup with the following shape. The cup is filled to the top.

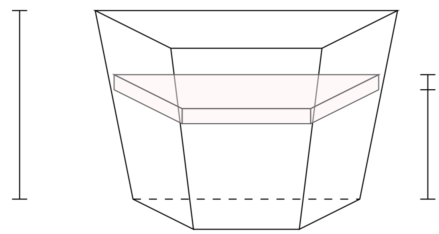


Figure 5.1: Cup

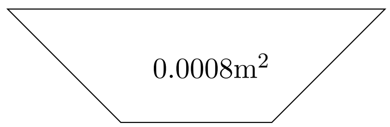


Figure 5.2: Top of cup

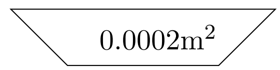


Figure 5.3: Bottom of cup

The area of a horizontal cross section of the cup (as shown in Figure 5.1) is **linear** with respect to h, its height above **the bottom of the cup.**
Flora is going to drink the juice with a magical straw. The top of the straw is always 0.05m above **the top of the cup**. Because the straw is magical, it extends automatically and the bottom end of the straw is always at the surface of the juice. The density of the juice is 1100kg/m^3. The gravitational acceleration is $g = 9.8\text{m/s}^2$.

a. [5 points] What is the approximate mass of the slice of juice that is h meters above **the bottom of the cup**, of thickness Δh meters (as shaded in Figure 5.1)? Do not simplify your answer. Include units.

Solution: First, calculate area of the slice at height h m. This is done by setting up linear equations. Let A be the area of the slice at height h.
When $h = 0$, $A = 0.0002$. When $h = 0.1$, $A = 0.0008$. Thus, slope of the linear equation is $\dfrac{0.0008 - 0.0002}{0.1 - 0} = 0.006$. By point-slope form,

$$A - 0.0002 = 0.006(h - 0),$$

$$A = 0.0002 + 0.006h.$$

The mass of the slice is then

$$\text{mass} = \text{Volume} \cdot \text{density} = (0.0002 + 0.006h)\Delta h \cdot 1100 \text{ kg}.$$

b. [3 points] What is the approximate work needed to lift the same slice of juice (h meters above **the bottom of the cup**, of thickness Δh meters, as shaded in Figure 5.1) to a height of 0.05m above **the top of the cup**? Do not simplify your answer. Include units.

Solution: Distance travelling for the slice is $0.05 + 0.1 - h$ m, so work for slice is

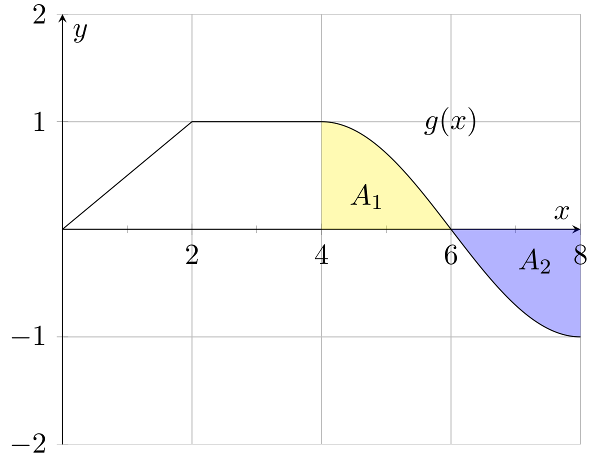
$$\text{Force} \cdot \text{distance travelling} = ((0.0002 + 0.006h)\Delta h \cdot 1100) \cdot g \cdot (0.05 + 0.1 - h) \text{ J.}$$

c. [2 points] Write an expression involving integrals for the total work needed to lift all the juice to a height of 0.05m above **the top of the cup**. Do not evaluate any integrals in your expression. Include units.

Solution:

$$\int_0^{0.1} ((0.0002 + 0.006h) \cdot 1100) \cdot g \cdot (0.05 + 0.1 - h)\, dh \text{ J.}$$

6. [15 points] Let $g(x)$ be an **odd** function, with part of the graph given as below.



Both shaded regions A_1 and A_2 have area 1.2.
Let $G(x)$ be an antiderivative of $g(x)$ with $G(2) = 2$.

a. [6 points] Copy the following table onto your paper and fill in with the exact values of $G(x)$. You do not need to show your work for this part, but you may receive credit for correct work shown.

x	-4	-2	0	2	4	6	8
$G(x)$				2			

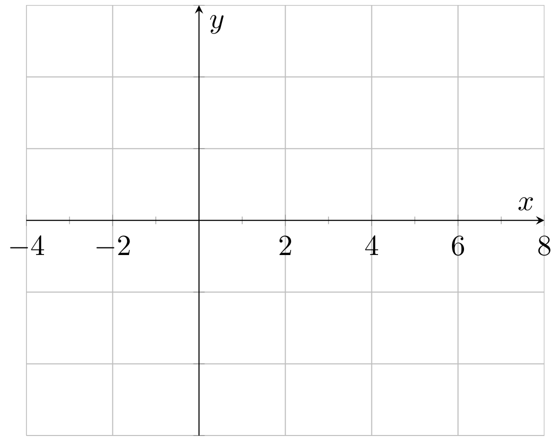
Solution:

x	-4	-2	0	2	4	6	8
$G(x)$	4	2	1	2	4	5.2	4

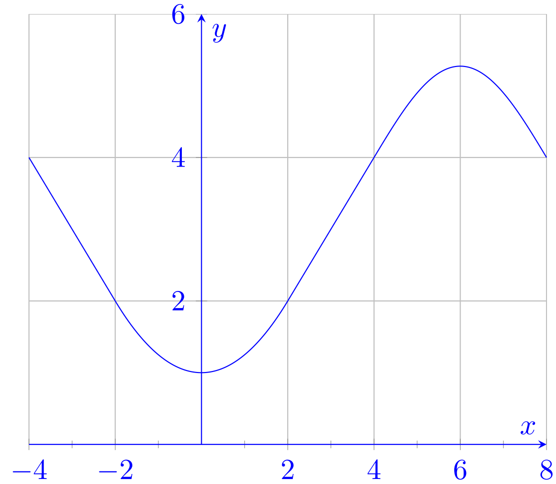
b. [9 points] Sketch a graph of $G(x)$ from $x = -4$ to $x = 8$ on **hand-drawn axes**, similar to those given below. Pay attention to

- if $G(x)$ is increasing / decreasing;
- if $G(x)$ is concave up / concave down / linear;
- all critical points and points of inflection.

Label the (x, y)-coordinates of all the critical points of $G(x)$. If you are worried that the concavity of your drawing is unclear, also label if each portion of your graph is concave up, concave down, or linear.



Solution:



7. [6 points] Split the function $\dfrac{5x^2-7x}{(x-1)^2(x+1)}$ into partial fractions with two or more terms. Do not integrate these terms. Be sure to show all work to obtain your partial fractions.

Solution: Let

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Solution 1: To solve for B: multiply both sides by $(x-1)^2$, then put $x=1$:

$$\frac{5x^2-7x}{x+1} = A(x-1) + B + C\frac{x-1}{x+1},$$

$$\frac{-2}{2} = 0 + B + 0, B = -1.$$

To solve for C: multiply both sides by $x+1$, then put $x=-1$:

$$\frac{5x^2-7x}{(x-1)^2} = A\frac{x+1}{x-1} + B\frac{x+1}{(x-1)^2} + C,$$

$$\frac{12}{4} = 0 + 0 + C, C = 3.$$

To solve for A, clear the denominator, and compare the coefficients.

$$5x^2 - 7x = A(x-1)(x+1) - (x+1) + 3(x-1)^2 = (A+3)x^2 - 7x + (2-A).$$

Say we look at the coefficients of x^2:

$$5 = A + 3, A = 2.$$

Therefore,

$$\frac{5x^2-7x}{(x-1)^2(x+1)} = \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{3}{x+1}.$$

Solution 2: Clear the denominator.

$$5x^2 - 7x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C).$$

Compare the coefficients.

$$A + C = 5, B - 2C = -7, -A + B + C = 0.$$

Solve the system of equations. An easy way is to write $A = 5 - C$, $B = 2C - 7$, and we then have $-(5-C) + (2C-7) + C = 0$. We have that $A = 2, B = -1, C = 3$.

8. [13 points] Let $f(x)$ be a twice differentiable function with

- $f(0) = 1$.

- $f(\ln 2) = \dfrac{5}{4}$.
- $f'(0) = e$.
- $f'(\ln 2) = 2$.

a. [3 points] Compute the average value of $f'(x)$ on $[0, \ln 2]$.

Solution: The average value is

$$\frac{1}{\ln 2 - 0}\int_0^{\ln 2} f'(x)\ dx = \frac{1}{\ln 2}(f(\ln 2) - f(0)) = \frac{1}{\ln 2}(\frac{5}{4} - 1).$$

b. [5 points] Compute the exact value of $\displaystyle\int_0^{\ln 2} xf''(x)\ dx$.

Solution:

$$\begin{aligned}\int_0^{\ln 2} xf''(x)\ dx &= \left(xf'(x)\right)_0^{\ln 2} - \int_0^{\ln 2} f'(x)\ dx \\ &= (\ln 2 f'(\ln 2) - 0f'(0)) - (f(\ln 2) - f(0)) \\ &= 2\ln 2 - \frac{5}{4} + 1.\end{aligned}$$

c. [5 points] Compute the exact value of $\displaystyle\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\ dx$.

Solution: Let $w = f(x)$. The new upper and lower bounds are $f(0) = 1$ and $f(\ln 2) = \dfrac{5}{4}$ respectively.

$$\int_0^{\ln 2} \frac{f'(x)}{\sqrt{9 - (f(x))^2}}\ dx = \int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w^2}}\ dw.$$

Let $w = 3\sin\theta$. The new θ-bounds are $\sin^{-1}\dfrac{1}{3}$ and $\sin^{-1}\dfrac{5}{12}$ respectively.

$$\begin{aligned}\int_1^{\frac{5}{4}} \frac{1}{\sqrt{9 - w}}\ dw &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} \frac{3\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\ d\theta \\ &= \int_{\sin^{-1}\frac{1}{3}}^{\sin^{-1}\frac{5}{12}} 1\ d\theta \\ &= \sin^{-1}\frac{5}{12} - \sin^{-1}\frac{1}{3}.\end{aligned}$$

9. [12 points] For each of the questions below, write out on your paper **all** the answers which are **always** true. For each answer you write, **give an explanation and/or a computation** that shows the statement is always true.

a. [6 points] Let $f(x)$ be a function defined for $0 \le x \le 1$ with $f(x) > 0$ and $f''(x) < 0$. Consider the Riemann sums for the integral $\int_0^1 f(x)\,dx$. Which of the following **must** be true?

LEFT(4) $\geq$ RIGHT(4)

MID(3) $\geq$ TRAP(2)

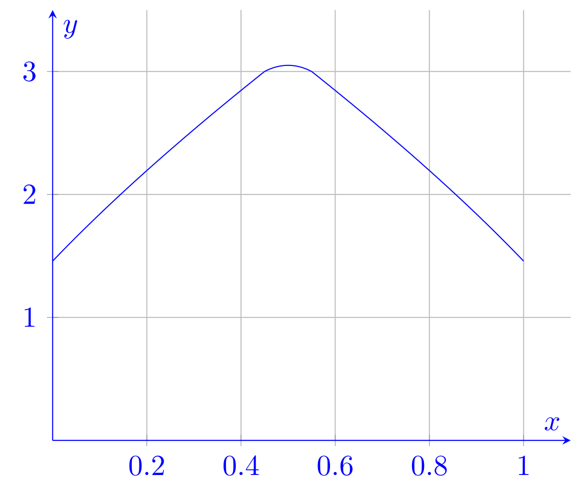
MID(3) is closer to the actual integral than MID(2) is

LEFT(3) $\leq$ TRAP(3)

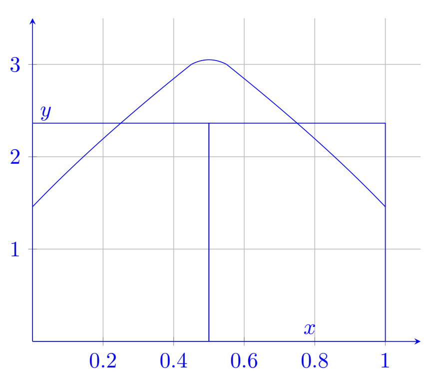
Solution: Only MID(3) $\geq$ TRAP(2) is always true. The function is concave down, so TRAP is an under-estimate and MID is an over-estimate. Hence MID(3) $\geq$ actual integral $\geq$ TRAP(2).

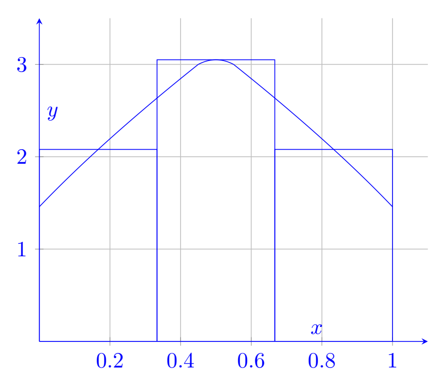
Extra explanation for why other choices are incorrect:
Choices involving LEFT or RIGHT: We do not know if $f(x)$ is increasing or decreasing, so we can't tell anything for LEFT or RIGHT.

Bottom left choice: In general, having more subdivisions does not guarantee that the estimates will always be better. What we can say is that the upper bound for the error will become smaller in the long run. However, in general it is always possible that a certain Riemann sum hits the exact value and some other Riemann sums with more subdivisions do not.
Even for concave up/down functions, it is possible that a certain MID with fewer subdivisions is better than another MID with more subdivisions. An actual graph of MID(2) better than MID(3) can be seen as below:



This graph is almost linear at $[0, 0.45]$ and $[0.55, 1]$ (you can see that it is still concave down if you zoom in a bit), and has a small hill around at $[0.45, 0.55]$. Since the graph is almost linear at $[0, 0.45]$, the first rectangle in MID(2) and the first rectangle in MID(3) is a good estimate of the respective area under the graph. Same applies to the last rectangles in MID(2) and MID(3). However, the middle rectangle of MID(3) is going to be overestimating by quite a bit. Therefore, MID(2) is a better estimate than MID(3).





b. [6 points] Let $g(x)$ be a differentiable function such that

- $0 < g(x) < 1$;
- $g'(x) \neq 0$.

Which of the following **must** be equal to $\dfrac{1}{\sin(g(x))}$?

$$\int_0^x \frac{g'(t)}{\sqrt{1-(g(t))^2}}\,dt \qquad \frac{d}{dx}\left(-\ln\left(\frac{1}{\sin(g(x))}+\frac{\cos(g(x))}{\sin(g(x))}\right)\right)$$

$$\frac{1}{g'(x)}\frac{d}{dx}\left(\int_1^{g(x)}\frac{1}{\sin t}\,dt\right) \qquad \int_0^x \frac{-g'(t)}{\sin(g(t))\tan(g(t))}\,dt + \frac{1}{\sin(g(0))}$$

Solution: $\dfrac{1}{g'(x)}\dfrac{d}{dx}\left(\displaystyle\int_1^{g(x)}\frac{1}{\sin t}\,dt\right)$ is correct, because

$$\frac{d}{dx}\left(\int_1^{g(x)}\frac{1}{\sin t}\,dt\right) = \frac{1}{\sin(g(x))}\cdot g'(x)$$

by chain rule.

$\displaystyle\int_0^x \frac{-g'(t)}{\sin(g(t))\tan(g(t))}\,dt + \frac{1}{\sin(g(0))}$ is correct. This can be seen by 2nd FTOC: this choice has the same values at $x = 0$ as $\dfrac{1}{\sin(g(x))}$ does, namely they are both $\dfrac{1}{\sin(g(0))}$. This choice also shares the same derivative as $\dfrac{1}{\sin(g(x))}$ does, namely

$$\frac{d}{dx}\frac{1}{\sin(g(x))} = \frac{-1}{\sin^2(g(x))}\cdot\cos(g(x))\cdot g'(x) = \frac{-g'(x)}{\sin(g(x))\tan(g(x))}.$$

Extra explanation for why other choices are incorrect:

$\displaystyle\int_0^x \frac{g'(t)}{\sqrt{1-(g(t))^2}}\,dt$ is not correct. It equals $\sin^{-1}(g(x))$, which is not the same as $\dfrac{1}{\sin(g(x))}$.

$\dfrac{d}{dx}\left(-\ln\left(\dfrac{1}{\sin(g(x))}+\dfrac{\cos(g(x))}{\sin(g(x))}\right)\right)$ is not correct, because this choice equals

$$\frac{1}{\sin(g(x))}\cdot g'(x).$$

A quick and easy way to rule this choice out (without actually calculating the derivative) is to notice that there will be a $g'(x)$ coming out from chain rule and no way to cancel it, but there is no $g'(x)$ in $\dfrac{1}{\sin(g(x))}$.