# Math 116 — First Midterm — March 29, 2021

1. You must write your solutions, including all work, for this exam on blank paper and upload you solutions to Gradescope when the exam is over. You may not write your answers on a printed copy of this exam.
2. Write your UMID on the upper right corner of every page you submit. Do not write your name on your submission.
3. This exam has 15 pages including this cover.
4. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any digital or computational device while working on this exam is not permitted. This means you may NOT use a calculator, tablet, computer or other device to do the problems on the exam.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You may use any physical notes or books. These may be handwritten or typed.
12. You must use the methods learned in this course to solve all problems.

| Problem | Points | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 12 | |
| 3 | 7 | |
| 4 | 7 | |
| 5 | 12 | |

| Problem | Points | Score |
|---|---|---|
| 6 | 12 | |
| 7 | 6 | |
| 8 | 9 | |
| 9 | 17 | |
| 10 | 12 | |
| Total | 100 | |

**1.** [6 points] This question is about exam policies. These are the same questions sent to you before the exam. The answer to each is either Yes or No. For each question, write the letter corresponding to the question and the entire word YES or the entire word NO. You do not need to rewrite the questions on your paper.

**a.** Are you allowed to use any resources, typed or handwritten, by accessing them on a digital device (e.g. computer, tablet, or phone)?

**b.** Are you allowed to refer to a physical copy of the textbook during the exam?

**c.** Are you allowed to use a calculator on the exam?

**d.** If you notice a mistake in one of your answers while you are scanning, is it okay to correct that mistake?

**e.** Can the penalty for cheating be a failing grade in the course?

**f.** Is 0.333333333 an exact answer to the equation $3x = 1$?

*Solution:*

**a.** No

**b.** Yes

**c.** No

**d.** No

**e.** Yes

**f.** No

**2**. [12 points] In order to build a settlement on the island, intruders start cutting down trees at the forest, cutting the trees into logs, and putting the logs in a pile. Let $A_n$ be the number of logs they have in the pile at noon on the $n$-th day. The intruders have 100 logs in the pile at noon on the first day (so $A_1 = 100$). Every day (between noon on one day and noon on the next day), the building team uses 10% of the logs in the pile, while the log-cutting team adds 20 logs to the pile immediately before noon.

**a**. [4 points] Find $A_2$ and $A_3$. **You do not need to simplify your answers.**

*Solution:*

$$A_2 = 100 \cdot 0.9 + 20$$
$$A_3 = (100 \cdot 0.9 + 20) \cdot 0.9 + 20 = 100 \cdot 0.9^2 + 20 + 20 \cdot 0.9$$

**b**. [5 points] Find a **closed form** expression for $A_n$. Closed form means your answer should not include ellipses or sigma notation, and should NOT be recursive. **You do not need to simplify your closed form answer.**

*Solution:* From observing the pattern from $A_1$, $A_2$ and $A_3$, we have

$$\begin{aligned} A_n &= 100 \cdot 0.9^{n-1} + (20 + 20 \cdot 0.9 + 20 \cdot 0.9^2 + \cdots + 20 \cdot 0.9^{n-2}) \\ &= 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9}. \end{aligned}$$

Note that the term $100 \cdot 0.9^{n-1}$ is not part of the geometric series. There are $n-1$ terms in the geometric series, so the exponent in the closed form is $n-1$.

**c**. [3 points] How many logs will the intruders have in the pile in the long run?

*Solution:*

$$\lim_{n\to\infty} 100 \cdot 0.9^{n-1} + 20 \cdot \frac{1 - 0.9^{n-1}}{1 - 0.9} = 0 + 20 \cdot \frac{1}{1 - 0.9} = 20 \cdot 10 = 200.$$

**3**. [7 points] Emily is a physics teacher. She is demonstrating the physics of rocket-launching with a water rocket in her class today. The water rocket weighs 3 lbs on the ground, and Emily launches it straight up to 10 ft above the ground. During the launch, the rocket's weight decreases at constant rate (in lbs/ft) as the water is ejected from the rocket. When it reaches 10 ft above the ground, the rocket weighs 1 lb.

**a**. [3 points] Calculate the weight of the rocket when it is a height $h$ ft above the ground. Include units.

*Solution:* Since the rocket's weight decreases at a constant rate in lb / ft, the rocket's weight is a linear function of its height above the ground, i.e. $h$.
At $h = 0$, the weight is 3lb. At $h = 10$, the weight is 1lb. Hence the slope of weight in terms of height is

$$\frac{1-3}{10-0} = \frac{-1}{5}.$$

By using point-slope form with $h = 0$ and weight $=$ 3lb, we have that

$$(\text{Weight at height } h) - 3 = \frac{-1}{5}(h-0),$$

$$\text{Weight at height } h = 3 - \frac{h}{5} \text{ lb.}$$

**b**. [4 points] Write an expression involving integrals for the total work required to propel the rocket from the ground to a height of 10 feet above the ground (as described above). Do not evaluate any integrals in your expression. Include units.

*Solution:* Work to lift the rocket from $h$ ft above ground to $h + \Delta h$ ft above ground is

$$(\text{Weight at height } h)(\text{lb}) \cdot (\text{distance travelling})(\text{ft}) = (3 - \frac{h}{5})\Delta h(\text{ftlb}).$$

Note that there is no $g = 9.8$, since lb is a unit for force. Hence total work is

$$\int_0^{10} (3 - \frac{h}{5})\, dh \text{ ftlb.}$$

**4.** [7 points] Some of Flora's friends are hurt while the intruders are building the settlement. Flora and Nile are trying their best to heal them. Suppose $p(x)$ is the probability density function for the number of weeks, $x$, it takes for everyone to recover after intruders appear.

$$p(x) = \begin{cases} c & \text{if } 0 < x \leq 1, \\ 2c & \text{if } 1 < x \leq 3, \\ 0 & \text{else.} \end{cases}$$

**a.** [2 points] Find $c$.

*Solution:*

$$\text{Area under full graph } = c \cdot (1-0) + 2c \cdot (3-1) = 5c.$$

Since $p(x)$ is a probability density function, area under full graph $= 1$, so $c = 1/5$.

**b.** [5 points] Let $W(x)$ be the cumulative distribution function for $p(x)$. Showing your work, give a piece-wise defined formula for $W(x)$ in the form given below.

$$W(x) = \begin{cases} \underline{\hspace{4cm}} & \text{if } x \leq 0, \\ \underline{\hspace{4cm}} & \text{if } 0 < x \leq 1, \\ \underline{\hspace{4cm}} & \text{if } 1 < x \leq 3, \\ \underline{\hspace{4cm}} & \text{if } x > 3. \end{cases}$$

*Solution:*

$$W(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ cx = \frac{x}{5} & \text{if } 0 < x \leq 1, \\ c + 2c(x-1) = -\frac{1}{5} + \frac{2}{5}x & \text{if } 1 < x \leq 3, \\ 1 & \text{if } x > 3. \end{cases}$$

Explanation:
There is no area under the pdf before $x = 0$, so $W(x) = 0$ if $x \leq 0$.
All area under the pdf is picked up after $x = 3$, so $W(x) = 1$ if $x > 3$.
For $0 < x \leq 1$,

$$W(x) = \int_0^x p(t)\ dt = cx.$$

For $1 < x \leq 3$,

$$W(x) = \int_0^x p(t)\ dt = \int_0^1 p(t)\ dt + \int_1^x p(t)\ dt = c + 2c(x-1).$$

**5**. [12 points] Another function $f(t)$ given by

$$f(t) = \begin{cases} \frac{t}{6} & \text{if } 0 < t \leq 2, \\ \frac{1}{3} & \text{if } 2 < t \leq 4, \\ 0 & \text{else.} \end{cases}$$

is the probability density function for the number of months $t$ that it will take the intruders to build the settlement.

**a**. [3 points] Find the probability that it will take the intruders between 1 and 2 months to build the settlement.

*Solution:* Probability that it will take the intruders between 1 and 2 months to build the settlement

$$= \int_1^2 f(t)\, dt = \int_1^2 \frac{t}{6}\, dt = \frac{2^2}{12} - \frac{1^2}{12} = \frac{3}{12} = \frac{1}{4}.$$

**b**. [5 points] Find the mean number of months it will take the intruders to build the settlement.

*Solution:*

$$\begin{aligned} \text{mean} &= \int_0^4 tf(t)\, dt \\ &= \int_0^2 tf(t)\, dt + \int_2^4 tf(t)\, dt \\ &= \int_0^2 \frac{t^2}{6}\, dt + \int_2^4 \frac{t}{3}\, dt \\ &= (\frac{2^3}{18} - \frac{0^3}{18}) + (\frac{4^2}{6} - \frac{2^2}{6}) \\ &= \frac{4}{9} + 2 \end{aligned}$$

**c**. [4 points] Find the median number of months it will take the intruders to build the settlement.

*Solution:* The area for the portion $2 < t \leq 4$ is $2/3$, which is more than $1/2$. Thus the median is between 2 and 4.
Let $T$ be the median. Then

$$\int_T^4 f(t)\ dt = \frac{1}{2},$$

$$(4-T)\frac{1}{3} = \frac{1}{2},$$

$$T = 4 - \frac{3}{2} = \frac{5}{2}.$$

**Alternatively:** We can set up with "the left half of the area is $1/2$". Let $T$ be the median.

$$\int_0^T f(t)\ dt = \frac{1}{2}.$$

With the same analysis as above, we know that the median lies between 2 and 4. Thus we split the integral.

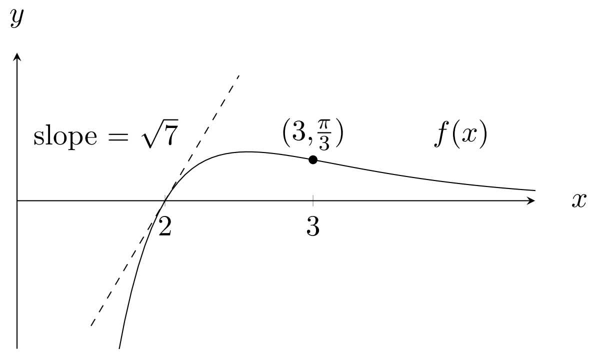
$$\int_0^2 f(t)\ dt + \int_2^T f(t)\ dt = \frac{1}{2},$$

$$\int_0^2 \frac{t}{6}\ dt + \int_2^T \frac{1}{3}\ dt = \frac{1}{2},$$

$$(\frac{2^2}{12} - \frac{0^2}{12}) + (\frac{T}{3} - \frac{2}{3}) = \frac{1}{2},$$

$$T = \frac{5}{2}.$$

**6**. [12 points] Below is the graph of a function $f(x)$. The function $f(x)$ is positive for $x > 2$. The $x$-axis is a horizontal asymptote of $f(x)$ as $x \to \infty$. The dashed line is the tangent line to $f(x)$ at $x = 2$, and its slope is $\sqrt{7}$.




**a**. [2 points] Compute $\displaystyle\lim_{x\to 2^+} \tan^{-1} \frac{1}{f(x)}$.

( *Hint:* Recall that $\displaystyle\lim_{z\to\infty} \tan^{-1} z = \frac{\pi}{2}$ and $\displaystyle\lim_{z\to-\infty} \tan^{-1} z = -\frac{\pi}{2}$. )

*Solution:* As $x \to 2^+$, $f(x)$ is approaching $0^+$ from the graph. Hence $\dfrac{1}{f(x)}$ is going to $+\infty$, so

$$\lim_{x\to 2^+} \tan^{-1} \frac{1}{f(x)} = \frac{\pi}{2}.$$

**b**. [3 points] Compute $\displaystyle\lim_{x\to 2^-} \frac{f(x)}{e^x - e^2}$.

*Solution:* As $x \to 2^-$, both the numerator and the denominator is approaching 0. Hence we can apply L'Hopital's Rule.

$$\lim_{x\to 2^-} \frac{f(x)}{e^x - e^2} = \lim_{x\to 2^-} \frac{f'(x)}{e^x} = \frac{\sqrt{7}}{e^2}.$$

**c**. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation.**

$$\int_3^\infty \frac{f'(x)}{(f(x))^{2/3}}\, dx$$

*Solution:*

$$\int_3^{\infty} \frac{f'(x)}{(f(x))^{2/3}}\ dx = \lim_{b\to\infty} \int_3^{b} \frac{f'(x)}{(f(x))^{2/3}}\ dx$$

Apply substitution, with $w = f(x)$.

$$\begin{aligned} &= \lim_{b\to\infty} \int_{f(3)}^{f(b)} \frac{1}{w^{2/3}}\ dw \\ &= \lim_{b\to\infty} 3w^{\frac{1}{3}}\Big|_{f(3)}^{f(b)} \\ &= \lim_{b\to\infty} 3((f(b))^{\frac{1}{3}} - (f(3))^{\frac{1}{3}}) \\ &= 3(0 - (\frac{\pi}{3})^{1/3}). \end{aligned}$$

**7.** [6 points] Determine whether the following improper integral converges or diverges. **Fully justify** your answer including using **proper notation**, and showing mechanics of any tests or theorems you use.

$$\int_0^1 \frac{\pi}{x^3+\sqrt{x}}\,dx$$

*Solution:* This integral is improper at the lower bound 0, since the denominator is 0 at $x=0$. Consider the dominating terms in the denominator. As $x \to 0^+$, $\sqrt{x}$ dominates $x^3$. As a result we should compare the integrand to $\pi/\sqrt{x}$.
In any case, since both $x^3$ and $\sqrt{x}$ are positive, if we take $x^3$ away, the denominator gets smaller. Hence the fraction gets bigger. Thus,

$$0 < \frac{\pi}{x^3+\sqrt{x}} \le \frac{\pi}{\sqrt{x}} \qquad \text{for } 0 < x \le 1.$$

The integral

$$\int_0^1 \frac{\pi}{\sqrt{x}}\,dx$$

converges by $p$-test, $p = 1/2$. Therefore, by comparison test,

$$\int_0^1 \frac{\pi}{x^3+\sqrt{x}}\,dx$$

converges.

**8**. [9 points] Consider the following 4 sequences.

$$(A)\quad a_n = (-1)^n, \qquad (B)\quad b_n = 3\cdot(0.5)^n,$$

$$(C)\quad c_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad (D)\quad d_n = \int_0^n \frac{x}{e^x}\,dx$$

For each of the following, write down the CAPITAL LETTER corresponding to each of the sequences that satisfy the given property. **No justification is required.**

**a**. [3 points] Which sequence(s) is/are bounded?

*Solution:* A,B,D

**b**. [3 points] Which sequence(s) is/are monotone?

*Solution:* B,C,D

**c**. [3 points] Which sequence(s) is/are convergent?

*Solution:* B,D

**9.** [17 points]

**a.** [4 points] Ivan is studying the series $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ and writes the following argument:

*The series is alternating. If we let $a_n = |(-1)^n(1+e^{-n})| = 1+e^{-n}$, then $a_n$ is positive, decreasing, but $\lim_{n\to\infty} a_n = 1$ is not 0. Therefore $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges by the alternating series test.*

Ivan's instructor tells Ivan that even though the $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ does diverge, the above argument is incorrect. Explain what's wrong with this argument, and give a correct argument to show that $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges.

*Solution:* Alternating series test can **only** show convergence. It should not be used to show divergence.
For this series, since $(-1)^n(1+e^{-n})$ oscillates near 1 and $-1$ as $n\to\infty$, the limit $\lim_{n\to\infty}(-1)^n(1+e^{-n})$ does not exist, in particular not 0. By $n$-term test, the series $\sum_{n=1}^{\infty}(-1)^n(1+e^{-n})$ diverges.

**b.** [6 points] Determine whether the following series converge or diverge. **Fully justify** your answer including using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!}$$

*Solution:* Since this series involves factorials, we apply ratio test.

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{(n+1)!}{(2n+3)!}\cdot\frac{(2n+1)!}{n!}\right|\\ &=\lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\cdot\frac{(2n+1)!}{(2n+3)!}\right|\\ &=\lim_{n\to\infty}\left|(n+1)\cdot\frac{1}{(2n+2)(2n+3)}\right|\\ &=0<1.\end{aligned}$$

Since the limit is $<1$, the series $\sum_{n=1}^{\infty}\frac{n!}{(2n+1)!}$ converges by ratio test.

**c**. [7 points] Determine whether the following series **converges absolutely, converges conditionally, or diverges**. Be sure to fully justify your answer, using **proper notation** and showing mechanics of any tests or theorems you use.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$$

*Solution:* We first see if the series converges absolutely by considering

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n(\ln n)^2} \right| = \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

To consider this series, we apply integral test. The function $f(x) = \frac{1}{x(\ln x)^2}$ is positive and decreasing for $x \geq 2$. As for the integral,

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\, dx.$$

Substitute with $w = \ln x$.

$$\begin{aligned} &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{w^2}\, dw \\ &= \lim_{b\to\infty} \left( \frac{-1}{w} \right)_{\ln 2}^{\ln b} \\ &= \lim_{b\to\infty} \left( -\frac{1}{\ln b} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}. \end{aligned}$$

Since the integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges by integral test. Hence the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges absolutely.

**10.** [12 points] Show that the following statements are false by giving a concrete example to contradict each of the statement. You can write a formula or draw a clear, well-labeled graph in place of the blanks. Accompany your example with a brief but complete explanation.

**a.** [4 points] If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Give your answer in the form:
"The statement is false when $a_n =$ ______________ because..."

*Solution:* For example, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by $p$-test, $p = 1$.

**b.** [4 points] For any continuous function $f(x)$ with $f(x) > 0$, the improper integral $\int_{-100}^{\infty} f(x)\, dx$ always diverges.

Give your answer in the form:

"The statement is false when $f(x) =$ ______________ because..."

*Solution:* An example is $f(x) = e^{-x}$, as $\int_{-100}^{\infty} e^{-x}\, dx = e^{100}$. We can also see that the integral converges by exponential decay test.

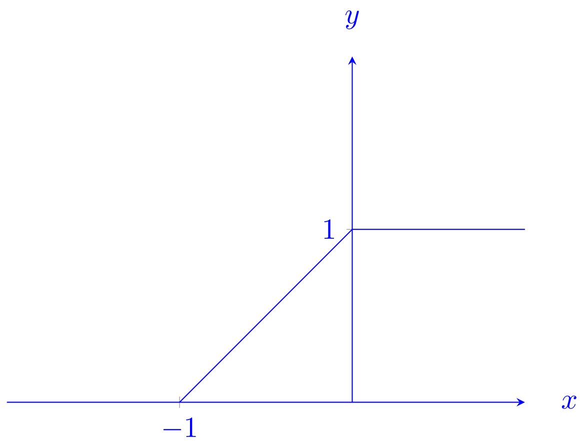
**c.** [4 points] If $P(x)$ is a cumulative distribution function, then $P(0) = 0$.

Give your answer in the form:

"The statement is false when $P(x) =$ ______________ because..."

(Note: Your $P(x)$ needs to be a cumulative distribution function, but you do not need to show/prove that it is.)

*Solution:* An example of $P(x)$ is given by the following graph.




In particular, $P(x)$ is indeed a cumulative distribution function, as $P(x)$ is increasing from 0 to 1 and it is continuous. However, $P(0) = 1 \neq 0$.