Math 116 — Final Exam — April 23, 2021

1. You must write your solutions, including all work, for this exam on blank paper and upload you solutions to Gradescope when the exam is over. You may not write your answers on a printed copy of this exam.
2. Write your UMID on the upper right corner of every page you submit. Do not write your name on your submission.
3. This exam has 15 pages including this cover.
4. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. The use of any digital or computational device while working on this exam is not permitted. This means you may NOT use a calculator, tablet, computer or other device to do the problems on the exam.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. **All** answers need to be in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You may use any physical notes or books. These may be handwritten or typed.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	7	
2	13	
3	9	
4	12	
5	5	

Problem	Points	Score
6	8	
7	15	
8	10	
9	3	
10	6	
11	12	
Total	100	

1. [7 points] This question is about exam policies. These are the same questions sent to you before the exam. The answer to each is either Yes or No. For each question, write the letter corresponding to the question and the entire word YES or the entire word NO. You do not need to rewrite the questions on your paper.

a. Are you allowed to use any resources, typed or handwritten, by accessing them on a digital device (e.g. computer, tablet, or phone)?

b. Are you allowed to refer to a physical copy of the textbook during the exam?

c. Are you allowed to use a calculator on the exam?

d. If you notice a mistake in one of your answers while you are scanning, is it okay to correct that mistake?

e. Can the penalty for cheating be a failing grade in the course?

f. Is 0.333333333 an exact answer to the equation $3x = 1$?

g. Must all answers on this exam be given in exact form?

Solution:

a. No

b. Yes

c. No

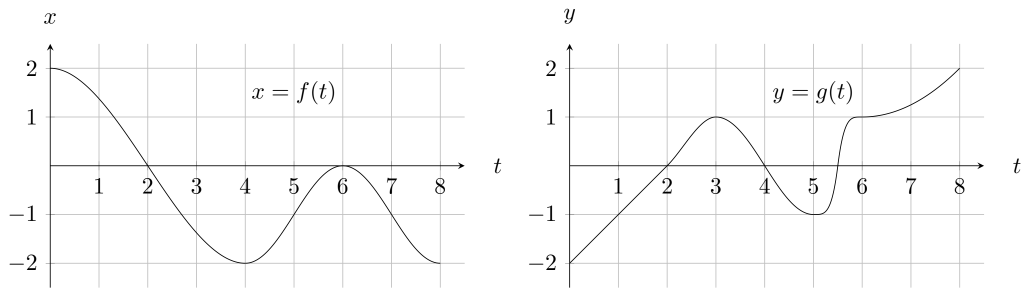
d. No

e. Yes

f. No

g. Yes

2. [13 points] To scare intruders off the island, Flora chases the intruders around. Her position at t minutes after she begins chasing the intruders is given by a parametric curve $(x, y) = (f(t), g(t))$. The graphs of $f(t)$ and $g(t)$ are given below, with x, y in km. For this question, "north" is the positive y-direction, and "east" is the positive x-direction.



a. [1 point] What is Flora's position at $t = 0$?

Solution: $f(0) = 2$ and $g(0) = -2$, so Flora is at $(2, -2)$ at $t = 0$.

b. [2 points] For $0 \le t \le 8$, at which t-value(s) is Flora at (0,0)? If there is no such time, write "NONE".

Solution: $f(t) = 0$ at $t = 2, 6$, and $g(t) = 0$ at $t = 2, 4, 5.5$, so Flora is at $(0, 0)$ only at $t = 2$.

c. [2 points] For $0 \le t \le 8$, at which t-value(s) is Flora going directly west (i.e. not in any northwest or southwest direction)? If there is no such time, write "NONE".

Solution: "Going west" means $f'(t) < 0$, and "going directly west" means Flora isn't travelling at any y-direction, i.e. $g'(t) = 0$. We have that $g'(t) = 0$ at $t = 3, 5, 6$. Among these 3 t-values, only at $t = 3$ is $f'(t) < 0$. So Flora is going directly west only at $t = 3$.

d. [2 points] For $0 \le t \le 8$, during which t-**interval(s)** is Flora going south? This includes any southeast and southwest directions, not only directly south. If there is no such time, write "NONE".

Solution: "Going south" means $g'(t) < 0$, so t needs to be in $(3, 5)$.

e. [2 points] For $0 \le t \le 8$, at which t-value(s) does Flora come to a stop? If there is no such time, write "NONE".

Solution: "Coming to a stop" means both $f'(t)$ and $g'(t)$ are 0. From the reasoning in (c), $g'(t) = 0$ when $t = 3, 5, 6$. Among these 3 points, only at $t = 6$ is $f'(t) = 0$. So Flora only comes to a stop at $t = 6$.

f. [4 points] Given that $f(1) = 4/3$, $f'(1) = -5/4$, and $g(t)$ is linear for $0 < t < 2$, find an equation for the tangent line to Flora's path at $t = 1$, given in Cartesian coordinates.

Solution: From the graph, $g(1) = -1$ and $g'(1) = 1$. The slope of the tangent line at $t = 1$ is

$$\frac{g'(1)}{f'(1)} = \frac{1}{-5/4} = -\frac{4}{5}.$$

By point-slope form, the tangent line at $t = 1$ is

$$y - (-1) = -\frac{4}{5}(x - \frac{4}{3}).$$

3. [9 points] Nile is also chasing the intruders, and her position t minutes after she starts chasing them is given by the following parametric curve (where units for both x and y are km).

$$x(t) = \frac{1}{t+1}, \qquad y(t) = t\cos t.$$

a. [4 points] Write, but do **not** evaluate, an integral to give the total distance travelled by Nile in the first minute after she started chasing the intruders.

Solution:

$$\int_0^1 \sqrt{\left(\frac{-1}{(t+1)^2}\right)^2 + (\cos t - t\sin t)^2}\, dt$$

b. [5 points] The intruders are scared and start to flee. Their total distance travelled (in km) is given by the integral

$$\int_1^\infty \sqrt{\frac{1}{u^2} + \frac{1}{e^u}}\, du$$

Does this improper integral converge or diverge? Fully justify your answer including using proper notation, and showing mechanics of any tests or theorems you use. Do not attempt to directly evaluate this integral.

Solution: For $u \geq 1$,

$$\sqrt{\frac{1}{u^2} + \frac{1}{e^u}} \geq \sqrt{\frac{1}{u^2}} = \frac{1}{u} > 0.$$

Since

$$\int_1^\infty \frac{1}{u}\, du$$

diverges by p-test, $p = 1$, the integral

$$\int_0^1 \sqrt{\frac{1}{u^2} + \frac{1}{e^u}}\, du$$

diverges by comparison test.

4. [12 points] Let

$$G(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} x^n.$$

a. [6 points] What is the **interval** of convergence for $G(x)$? Show the mechanics of any tests or theorems you use. Take as given, and **do not show**, that the radius of convergence of $G(x)$ is 5.

Solution: The center is at $x = 0$, so the two end points are 5 and -5.
At $x = 5$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} 5^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Since $\frac{1}{n}$ is positive, decreasing, and $\lim_{n\to\infty} \frac{1}{n} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by alternating series test.
At $x = -5$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}(-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n}(-1)^n 5^n = -\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges by p-test, $p = 1$.
Therefore, the interval of convergence is $(-5, 5]$.

b. [3 points] Find $G^{(100)}(0)$.

Solution: Consider the coefficient of x^{100} of the power series.

$$\frac{G^{(100)}(0)}{100!} = \frac{(-1)^{101}}{100 \cdot 5^{100}},$$

$$G^{(100)}(0) = -\frac{100!}{100 \cdot 5^{100}}.$$

c. [3 points] Compute the exact value of $G(2)$.

Solution:

$$G(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n5^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{2}{5}\right)^n.$$

From the list of "Known" Taylor series,

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

for $-1 < x \le 1$. Plug in $x = 2/5$, we have that $G(2) = \ln(1 + \frac{2}{5}) = \ln(7/5)$.

5. [5 points] Let $p(x)$ be the probability density function for the price of a meal on South University Avenue where x is given in dollars. The formula of $p(x)$ is given as follow:

$$p(x) = \frac{1}{\sqrt{\pi}} e^{-(x-9)^2}$$

a. [2 points] Write, but do **not** evaluate, an integral that gives the probability of a meal on South University Avenue being between \$8 and \$14.

Solution:

$$\int_8^{14} \frac{1}{\sqrt{\pi}} e^{-(x-9)^2}\, dx.$$

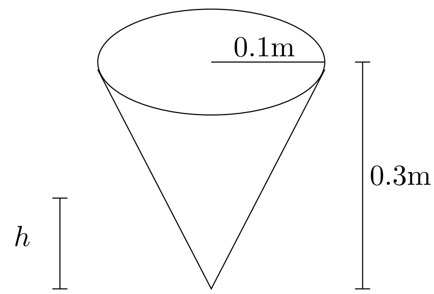
b. [3 points] Write, but do **not** simplify, an expression that estimates your integral in (a) by MID(3). Be sure to write out all the terms in your sum.

Solution: We have $\Delta x = (14-8)/3 = 2$, so the subdivisions are: 8 to 10, 10 to 12, 12 to 14. The data points for the MID sum are 9,11,13, and

$$\text{MID}(3) = 2 \cdot \frac{1}{\sqrt{\pi}} (e^{-(9-9)^2} + e^{-(11-9)^2} + e^{-(13-9)^2}).$$

6. [8 points] Ari and Bell are enjoying their time at a beach.

a. [5 points] Ari has an ice cream cone of radius 0.1m and height 0.3m, as shown in the following picture. The cone is filled to the top with ice cream, and the ice cream located a vertical distance h meters above the bottom tip of the cone (the point at the bottom of the figure below) has density $\delta(h) = \ln(2-h)$ kg / m^3. An example of the vertical distance h is shown in the figure below.



Write, but do **not** compute, one or more integral(s) to express the total mass of the ice cream cone. Include units.

Solution: By similar triangles or setting up linear equations, radius at vertical distance h m above the bottom tip is given by

$$\frac{r}{0.1} = \frac{h}{0.3}, \qquad r = \frac{h}{3}.$$

Mass of the slice of ice cream at height h m is

$$\pi\left(\frac{h}{3}\right)^2 \Delta h \ln(2-h) \text{ kg}.$$

Total mass of ice cream is

$$\int_0^{0.3} \pi\left(\frac{h}{3}\right)^2 \ln(2-h)\, dh \text{ kg}.$$

b. [3 points] Bell is lifting a bottle of water straight upwards 3 meters at a constant speed. The bottle initially has a mass of 2kg, and it is leaking at a steady rate of 0.5 kg / m. Assume that gravitational acceleration is $g = 9.8$ m / s^2. Write, but do **not** compute, one or more integral(s) to express the total work done by Bell on the bottle. Include units.

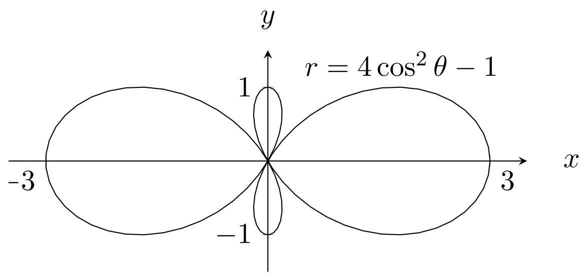
Solution: Let h m be the distance of the bottle of water from its starting location at a particular instance. The mass of water at the position is $2 - 0.5h$ kg. The slice of work to lift the bottle from this position for Δh meter is given by

$$(2 - 0.5h) \cdot g \cdot \Delta h \text{ J}.$$

Total work is

$$\int_0^3 (2 - 0.5h) \cdot g\, dh \text{ J}.$$

7. [15 points] Nat is sailing a boat in a lake, with the path given by the following polar graph.



a. [4 points] What are all the angles θ, with $0 \leq \theta \leq 2\pi$, for which the graph passes through the origin?

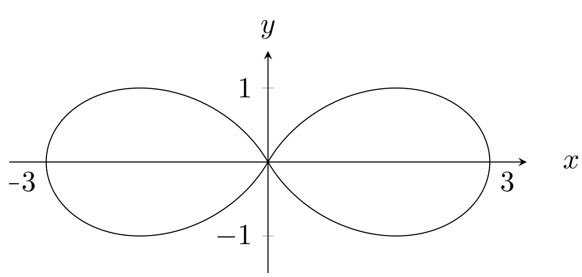
Solution: At the origin, $r = 0$.

$$4\cos^2\theta - 1 = 0$$
$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -\frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{or} \quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

So $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

b. [4 points] Write down, but do **not** evaluate, one or more integral(s) that gives the arc length of the larger horizontal figure 8 from the path above, as given in the following graph.



Solution: When $\theta = 0$, we have $r = 4 - 1 = 3$, so the x, y-coordinates of the point is $(3, 0)$. When θ increases, the graph travels counter-clockwise. The part in the 1st quadrant is traced when θ increases from $\theta = 0$ til the first time the graph meets the origin, i.e. $\pi/3$. As for the 4th quadrant, the portion of the graph is traced when θ decreases from $\theta = 0$ til the first time the graph meets the origin at a negative angle, i.e. $\theta = 5\pi/3 - 2\pi = -\pi/3$. Therefore, the right loop has arc length

$$\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

By symmetry, the total arc length of the horizontal figure 8 is given by doubling the right loop, i.e.

$$2\int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

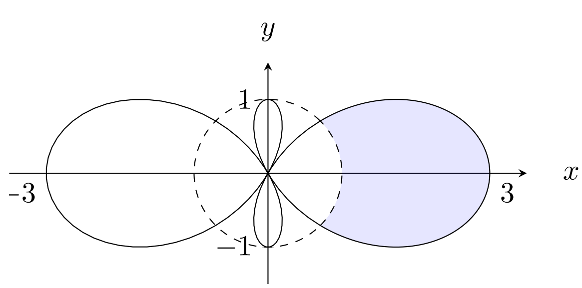
Alternatively, the left loop is traced from $\theta = 2\pi/3$ to $\theta = 4\pi/3$, so the total arc length can also be written as

$$\begin{aligned}\text{left loop} + \text{right loop} &= \int_{2\pi/3}^{4\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta \\ &\quad + \int_{-\pi/3}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.\end{aligned}$$

Alternatively, the total arc length of the horizontal figure 8 is 4 times the arc length of the portion in the first quadrant, by symmetry, so it is

$$4\int_{0}^{\pi/3} \sqrt{(4\cos^2\theta - 1)^2 + (-8\cos\theta\sin\theta)^2}\, d\theta.$$

c. [5 points] Another boat is travelling around the unit circle $r = 1$, given by the dashed curve in the graph below. Write down, but do **not** evaluate, one or more integral(s) that gives the area of the shaded region, as shown below.



Solution: The two graphs intersect once at $0 < \theta < \pi/3$, and another time at $-\pi/3 < \theta < 0$. To solve for the angle of intersection, we set the two r to be equal.

$$4\cos^2\theta - 1 = 1$$

$$\cos^2\theta = \frac{1}{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}} \qquad \text{or} \qquad \cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}, \frac{7\pi}{4} \qquad \text{or} \qquad \theta = \frac{3\pi}{4}, \frac{5\pi}{4},$$

or adding integer multiples of 2π to any of these.
We want an angle in $0 < \theta < \dfrac{\pi}{3}$ and another in $-\dfrac{\pi}{3} < \theta < 0$, so we have $\theta = \dfrac{\pi}{4}$ and $-\dfrac{\pi}{4}$.
Therefore, the area bounded by the solid curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2\, d\theta.$$

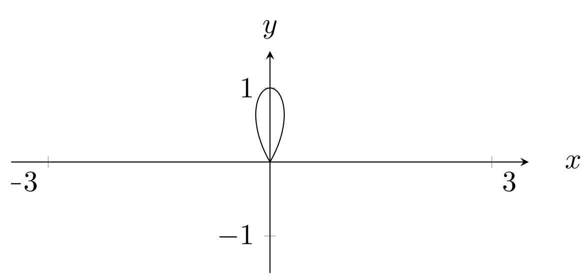
The area bounded by the dashed curve is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 1^2\, d\theta = \frac{\pi}{4},$$

so the shaded area is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (4\cos^2\theta - 1)^2 - 1^2\, d\theta.$$

d. [2 points] Give an interval of θ-values for which the polar equation $r = 4\cos^2\theta - 1$ traces out the upper loop of the smaller figure 8 as shown below.



Solution: From the analysis in (b), we know that as θ increases from 0 to $\pi/3$, the first quadrant portion of the horizontal figure 8 is traced out.
Then as θ increases from $\pi/3$ to $2\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. As a result, the direction of the point is **opposite** to the direction indicated by the angle θ. This means that we are below the x-axis instead of above the x-axis. As a result, the **lower** half of the small loop is traced out in $[\pi/3, 2\pi/3]$.
When θ increase from $2\pi/3$ to $4\pi/3$, the formula $r = 4\cos^2\theta - 1$ stays positive. Hence the direction is given by the angle indicated by the angle θ. At this interval, the left half of the big horizontal figure 8 is traced out.
Finally, when θ increases from $4\pi/2$ to $5\pi/3$, we have that $r = 4\cos^2\theta - 1$ is negative. Hence we need to go to the opposite direction as indicated by θ. As a result, we are at the **upper** half of the small loop when θ is in $[4\pi/3, 5\pi/3]$.

8. [10 points] For each of the questions below, write out on your paper **all** the answers which are **always** true. No explanation is needed.

a. [3 points] Given that the power series $\sum_{n=0}^{\infty} C_n(x-1)^n$ converges at $x=3$ and diverges at $x=8$, at which of the following x-value(s) **must** the series **converge**?

-7 $\quad -6$ $\quad -3$ $\quad -1$ $\quad 0$ $\quad 2$ $\quad 6$ $\quad 9$ $\quad$ NONE OF THESE

Solution: 0,2
The power series converges at $x=3$, which is 2 away from the center $x=1$. Thus the radius of convergence is at least 2, so the power series converges at $x=0,2$, which are within 2 from the center.
The power series does not need to converge at $x=-1$ (also 2 away from the center), since $x=-1$ could potentially be an end point of the interval of convergence.

b. [3 points] Note: This part has the same set up as (a), but asks about divergence.

Given that the power series $\sum_{n=0}^{\infty} C_n(x-1)^n$ converges at $x=3$ and diverges at $x=8$, at which of the following x-value(s) **must** the series **diverge**?

-7 $\quad -6$ $\quad -3$ $\quad -1$ $\quad 0$ $\quad 2$ $\quad 6$ $\quad 9$ $\quad$ NONE OF THESE

Solution: -7,9
The power series diverges at $x=8$, which is 7 away from the center $x=1$. Thus the radius of convergence is at most 7, so the power series diverges at $x=-7,9$, which are more than 7 away from the center.
The power series does not need to converge at $x=-6$ (also 7 away from the center), since $x=-6$ could potentially be an end point of the interval of convergence.

c. [4 points] Let $x=f(t)$, $y=g(t)$ (where $0\le t\le 10$) be a parametric curve such that $y=x^2$. Which of the following must be true?

(i) If V is the **speed** of the curve at $t=4$, then $V\ge f'(4)$.

(ii) $f'(t)\ge 0$ for $0<t<10$.

(iii) $g(t)\ge 0$ for $0<t<10$.

(iv) The tangent line to the curve at $t=1$ is $y=2x-1$.

(v) NONE OF THE ABOVE

Solution: (i), (iii)
(i): $V=\sqrt{(f'(4))^2+(g'(4))^2}$. Since $(g'(4))^2\ge 0$, we have $V\ge\sqrt{(f'(4))^2}=|f'(4)|$.
(ii): It is possible for $f'(t)<0$. For example, we can have $(x,y)=(-t,t^2)$.
(iii): $g(t)=y=x^2\ge 0$.
(iv): $y=2x-1$ is the tangent line to the curve at $(1,1)$. However, the curve is not necessarily at $(1,1)$ when $t=1$. For example, if $(x,y)=(-t,t^2)$, then the curve is at $(-1,1)$ when $t=1$.

9. [3 points] For $x > 0$, let $g(x)$ be a positive continuous function, and

$$G(x) = \int_x^{e^{x^2}} \frac{1}{g(t)}\, dt.$$

Find $G'(x)$. Your answer may involve g.

Solution:

$$G'(x) = \frac{1}{g(e^{x^2})} \cdot e^{x^2} \cdot 2x - \frac{1}{g(x)}.$$

10. [6 points] Compute the **radius** of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8^n}{(n+1)^2} x^{3n+1}$$

Be sure to show all your reasoning.

Solution: Use ratio test.

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{8^{n+1}x^{3n+4}}{(n+2)^2} \cdot \frac{(n+1)^2}{8^n x^{3n+1}} \right| &= \lim_{n\to\infty} \left| \frac{8^{n+1}}{8^n} \cdot \frac{x^{3n+4}}{x^{3n+1}} \cdot \frac{(n+1)^2}{(n+2)^2} \right| \\ &= \lim_{n\to\infty} \left| 8x^3 \cdot \frac{(n+1)^2}{(n+2)^2} \right| \\ &= 8|x|^3\end{aligned}$$

To have the power series converge, we need $8|x|^3 < 1$, i.e.

$$|x|^3 < \frac{1}{8}, \qquad |x| < \frac{1}{2}.$$

So the radius of convergence is $\frac{1}{2}$.

11. [12 points] Let $f(x) = x(1-x)^{-1/2}$.

a. [4 points] Write down the first 3 non-zero terms of the Taylor series for $f(x)$ centered at $x = 0$. Show your work.

Solution: From the list of "Known" Taylor series:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$$

Put $p = -\frac{1}{2}$:

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Replace x by $-x$:

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Multiply by x:

$$x(1-x)^{-1/2} = x + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \cdots$$

b. [3 points] Let $F(x)$ be an antiderivative of $f(x)$ such that $F(0) = 2$. Write down the first 4 non-zero terms of the Taylor series for $F(x)$ centered at $x = 0$. Show your work.

Solution: By the second fundamental theorem of calculus,

$$F(x) = 2 + \int_0^x f(t)\, dt.$$

Therefore,

$$F(x) = 2 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \cdots$$

c. [5 points] Compute the exact value of $\displaystyle\int_0^{3/4} f(x)\, dx$. Show each step of your computation.

Solution: Integration by parts: Let $u = x$ and $v' = (1-x)^{-1/2}$. Then $u' = 1$ and $v = -2(1-x)^{1/2}$. We have

$$\begin{aligned}\int_0^{3/4} x(1-x)^{-1/2}\, dx &= -2x(1-x)^{1/2}\Big|_0^{3/4} + 2\int_0^{3/4}(1-x)^{1/2}\, dx \\ &= \left(-2x(1-x)^{1/2} - \frac{4}{3}(1-x)^{3/2}\right)_0^{3/4} \\ &= (-\frac{3}{4} - \frac{1}{6}) + \frac{4}{3} = \frac{5}{12}.\end{aligned}$$

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$