Math 116 — Final Exam — April 22, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ____________________ Section #: ______

1. This exam has 16 pages including this cover.
2. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	11	
3	16	
4	8	
5	16	
6	9	

Problem	Points	Score
7	6	
8	11	
9	10	
Total	100	

1. [13 points] Universe of Movies (UofM) is a new online movie rating database, which assigns movies a rating from zero to five stars. The star ratings for each movie are not necessarily integer values. The following **probability density function** gives the distribution of star rating, t, for the films in the UofM database.

$$p(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{6} & 0 < t \leq 3 \\ \frac{t}{32} & 3 < t \leq 5 \\ 0 & 5 < t \end{cases}$$

a. [6 points] Write a formula for the corresponding cumulative distribution function, $F(t)$.

$$F(t) = \begin{cases} \underline{\hspace{4cm}} & t \leq 0 \\ \underline{\hspace{4cm}} & 0 < t \leq 3 \\ \underline{\hspace{4cm}} & 3 < t \leq 5 \\ \underline{\hspace{4cm}} & 5 < t \end{cases}$$

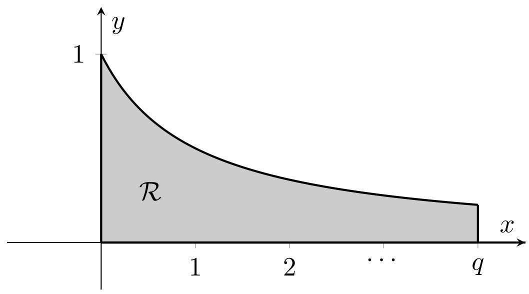
b. [3 points] What is the median number of stars for the films in the online database? Be sure to show any work.

Answer: ______________________

c. [4 points] Write, but do not compute, the formula for the mean number of stars for the movies in the UofM database. Write out formulas of all functions you use, i.e. do not include p or F in your answers.

Answer: ______________________

2. [11 points] Consider the function $f(x) = e^{-2x}$, and the region $\mathcal{R}$ bounded by the x-axis, the y-axis, $y = f(x)$ and $x = q$, where q is a positive constant larger than 2.



a. [4 points] Give a formula for, but do not compute, the volume of the solid formed by rotating the region $\mathcal{R}$ around the y-axis. Your answer should depend on q. *(Hint: Use the shell method)*

b. [4 points] Compute the integral you found in part a). Your final answer should be in terms of q.

c. [3 points] Taking a limit of your answer in b), compute the volume of the infinitely long solid of revolution formed by rotating the region $\mathcal{R}$ around the y-axis. Be sure to show how you got the value of your limit.

3. [16 points] Use the table below to answer the following questions. Write your answers using **exact form** on the blank provided. If there is not enough information to complete the problem, write N.I. You need to evaluate all integrals completely, but you do not need to simplify your final answers.

t	0	1	2	3	4	5
$f(t)$	3	2	5	e	7	0
$f'(t)$	-1	-4	3	2	π	1
$f''(t)$	2	8	1	2	6	2

a. [4 points] MID(2) for $\int_1^5 f(t)dt$

Answer: ______________________.

b. [4 points] TRAP (2) for $\int_0^4 tf'(t)dt$

Answer: ______________________.

c. [4 points] Average value of $f'(2t)$ on $[0, 2]$

Answer: ______________________.

d. [4 points] Second degree Taylor Polynomial of $f(-x)$ around $x = -2$.

Answer: ______________________.

4. [8 points] Suppose a_n and b_n are sequences of positive numbers, defined for $n = 1, 2, 3 \ldots$, satisfying the following conditions:

- $a_n \leq \frac{1}{n^{1/2}}$
- $b_n \geq \frac{1}{n}$

For each statement below, circle ALWAYS if the statement is always true, SOMETIMES if the statement can be true or false depending on the specifics of a_n or b_n, and NEVER if the statement is false for all specific a_n or b_n.

a. [1 point] $\lim_{n\to\infty} a_n = 0$.

ALWAYS SOMETIMES NEVER

b. [1 point] $\lim_{n\to\infty} b_n = 0$.

ALWAYS SOMETIMES NEVER

c. [1 point] a_n is bounded.

ALWAYS SOMETIMES NEVER

d. [1 point] b_n is monotone.

ALWAYS SOMETIMES NEVER

e. [1 point] $\sum_{n=1}^{\infty} a_n$ converges.

ALWAYS SOMETIMES NEVER

f. [1 point] $\sum_{n=1}^{\infty} b_n$ converges.

ALWAYS SOMETIMES NEVER

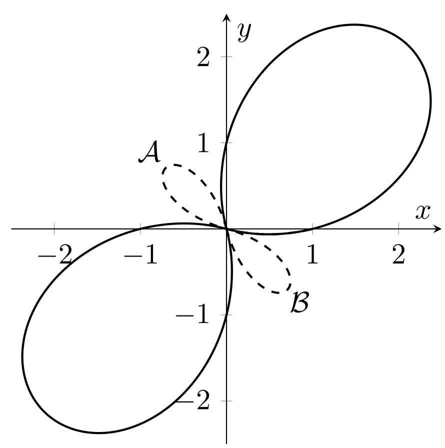
g. [1 point] $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

ALWAYS SOMETIMES NEVER

h. [1 point] $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

ALWAYS SOMETIMES NEVER

5. [16 points] The following problems relate to the polar graph shown below, defined by the polar curve $r(\theta) = 2\sin(2\theta) + 1$, on the domain $[0, 2\pi]$. Both the dashed and solid curves are part of the graph of $r(\theta)$.



a. [4 points] Find all θ values in the interval $[0, 2\pi]$ such that $r(\theta) = 0$.

Answers: ______________________

b. [4 points] Determine the θ intervals corresponding to the dashed portions $\mathcal{A}$ and $\mathcal{B}$ of the curve above.

Interval for $\mathcal{A}$: ______________ Interval for $\mathcal{B}$: ______________

c. [4 points] Write an expression involving one or more integrals for the area of the region enclosed by the **solid** curves only (do not include the region enclosed by the dashed curves).

d. [4 points] Write an expression involving one or more integrals for the total arc length of the **dashed** curves in the graph above.

6. [9 points] Suppose the Taylor series for a function $f(x)$ around $x = 3$ is

$$\sum_{n=1}^{\infty} \frac{(6)^{-n}((2n)!)}{n!(n-1)!}(x-3)^{2n}$$

a. [6 points] Compute the radius of convergence for this series. Be sure to fully justify your answer and show all work. Do not compute the interval of convergence.

Radius of Convergence: ____________________

b. [3 points] Find $f^{(2022)}(3)$ and $f^{(2023)}(3)$.

$f^{(2022)}(3)=$ ____________________ $f^{(2023)}(3)=$ ____________________

7. [6 points] Compute the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(x-4)^n$$

Show all work and give all necessary justification. You may assume it has radius of convergence 4 and you do not need to show this.

Interval of Convergence: ____________________

8. [11 points]

a. [6 points] Give the first three non-zero terms of the Taylor Series for the function:

$$(x^2 + 2)\sin(x)$$

centered at $x = 0$.

b. [5 points] Compute the limit:

$$\lim_{x\to 0} \frac{\int_0^{2x} \left(\left(\frac{t}{2}\right)^2 + 2\right) \sin\left(\frac{t}{2}\right) dt}{x^2}$$

Answer: ____________________

9. [10 points] After a mistake on their last mission, Brad and Angelina must go on the run together, or risk being captured by an opposing agent. Brad and Angelina's shared position is given by the parametric equations:

$$(f(t), g(t)) = (t^2 + 10,\ 2t^2 + 10)$$

and the agent pursuing them has position given by the equations

$$(r(t), q(t)) = (7t,\ \sin(\pi t) + 12t).$$

The time t is measured in hours after Brad and Angelina have gone on the run, and all distances are given in miles.

a. [5 points] The agent catches up with Brad and Angelina at the smallest positive t-value when the agent is in the same position as they are. Find the time when the agent catches up with Brad and Angelina.

Answer: ____________________

b. [5 points] Compute the total distance traveled by Brad and Angelina before the agent catches up with them.

Answer: ____________________

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n\, x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n\, x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}\, x^2 + \frac{p(p-1)(p-2)}{3!}\, x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$