Math 116 — First Midterm — February 7, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ____________________ Section #: ____

1. This exam has 14 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
7. Include units in your answer where that is appropriate.
8. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	16	
2	10	
3	12	
4	11	
5	13	
6	13	

Problem	Points	Score
7	13	
8	12	
Total	100	

1. [16 points] Use the table to compute the following integrals. Write your answer using exact form on the blank provided. If there is not enough information available to answer the question, write N.I. You need to evaluate all integrals, but you do not need to simplify your final answer.

x	1	1	2	3	π	4	8	9
$f(x)$	0	2	5	9	-3	π	6	4
$f'(x)$	-7	-3	4	7	2	1	0	-5
$g(x)$	5	3	-1	2	6	2	-3	e

a. [4 points] $\displaystyle\int_0^4 x^{\frac{1}{2}} g'(\sqrt{x^3})dx$

Answer: $\dfrac{-16}{3}$.

Solution: Let $u = x^{\frac{3}{2}}$, $\frac{2}{3}du = x^{\frac{1}{2}}dx$ so the integral becomes $\displaystyle\frac{2}{3}\int_0^8 g'(u)du = \frac{2}{3}(g(8) - g(0)) = \frac{2}{3}(-3-5) = \frac{-16}{3}$

b. [4 points] $\displaystyle\int_4^8 x f''(x)dx$

Answer: $-10+\pi$.

Solution: Let $u = x$ and $dv = f''(x)dx$, so we get $du = dx$ and $v = f'(x)$. Then we get $\displaystyle\int_4^8 x f''(x)dx = xf'(x)\Big|_4^8 - \int_4^8 f'(x)dx$, which gives $8f'(8) - 4f'(4) - f(8) + f(4) = 8(0) - 4(1) - 6 + \pi = -10 + \pi$

c. [4 points] $\displaystyle\int_0^{\pi} \cos(t) f(\sin(t))dt$

Answer: **0**.

Solution: Using substitution, we let $u = \sin(t)$, and so $du = cos(t)dt$. Then, as $\sin(\pi) = 0$, our integral becomes $\displaystyle\int_0^0 f(u)du = 0$.

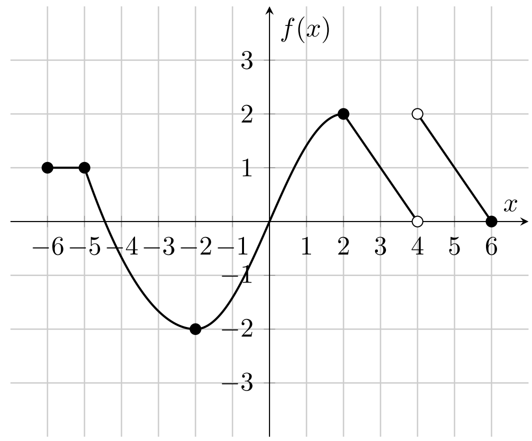
d. [4 points] $\displaystyle\int_4^9 f''(\sqrt{x})dx$

Answer: 18.

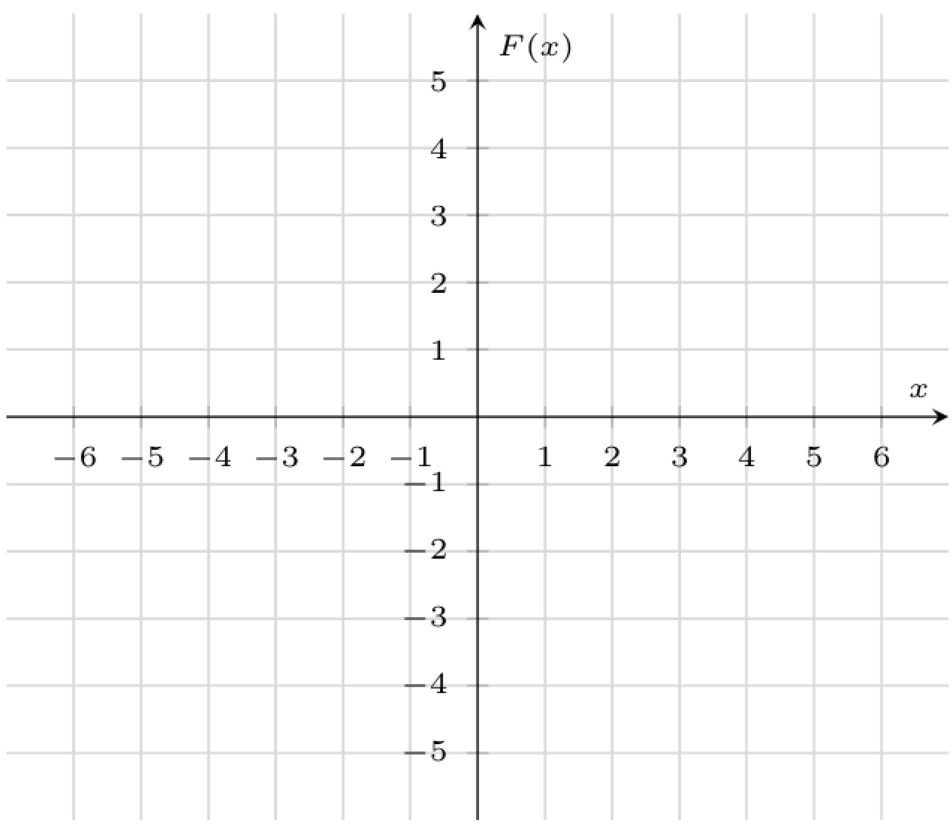
Solution: Let $u = \sqrt{x}$, and $du = \frac{1}{2\sqrt{x}}dx$. By noting that $\frac{1}{\sqrt{x}} = \frac{1}{u}$, we get that $2udu = dx$, and so integral becomes $\displaystyle 2\int_2^3 uf''(u)du$. Then, using integration by parts, our answer is $\displaystyle 2\left(uf'(u)\Big|_2^3 - \int_2^3 f'(u)du\right)$. The answer is then

$$2(3f'(3) - 2f'(2) - f(3) + f(2)) = 2(3(7) - 2(4) - 9 + 5) = 18$$

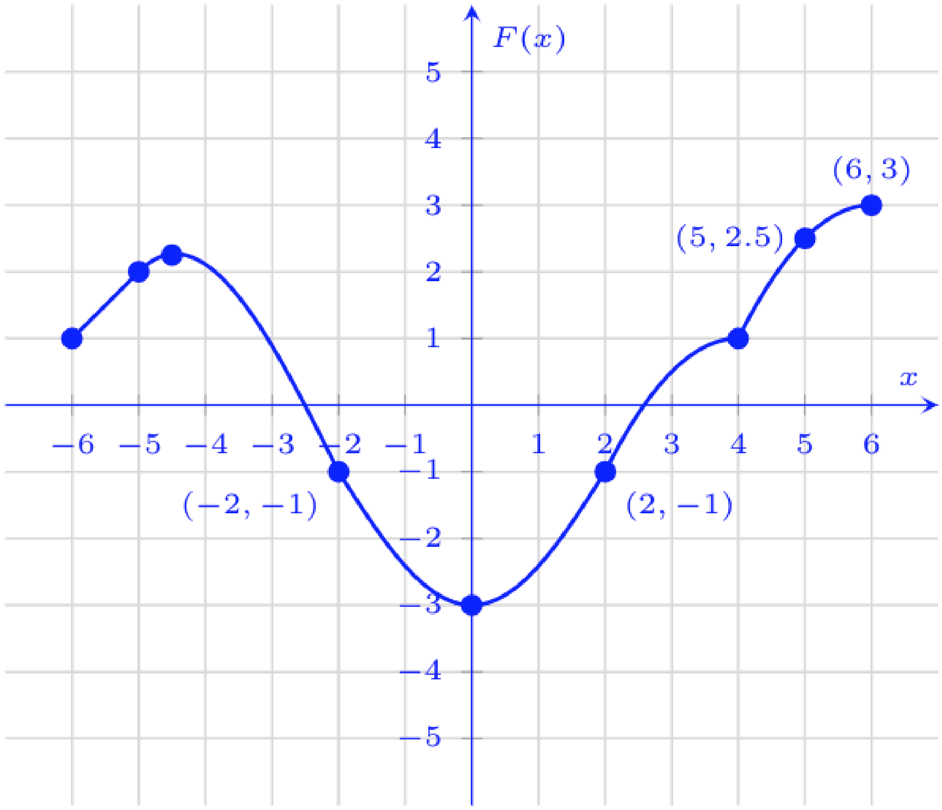
2. [10 points]



Sketch a continuous antiderivative $F(x)$ to the function $f(x)$ graphed above, such that $F(2) = -1$. The function $f(x)$ is **odd** on $[-2, 2]$. Make sure to clearly label the input and output values at $x = -2, 2, 5$, and 6. Be sure to make it clear where the graph is concave up, concave down, or linear, and where it is increasing or decreasing.



Solution:
The required labeled values are included in the graph below, and the value at $x = -6$ should be one less than the value at $x = -5$. The function should be increasing on $[-6, -4.5]$ and $[0, 6]$, decreasing on $[-4.5, 0]$, with maximum and minimum at the transition points (Near -4.5 is sufficient, as there is no way to determine this exactly). The graph should be concave down on $[-6, -2]$, $[2, 4]$ and $[4, 6]$, and concave up on $[-2, 2]$. The graph should level off approaching 4 from the left, as the graph of the derivative is approaching zero.



3. [12 points]

a. [6 points] Split the following function into partial fractions. Do not integrate the result. Be sure to show all your work.

$$\frac{10x+2}{(x-1)^2(x+2)}$$

Solution: Start by splitting:

$$\frac{10x+2}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

By giving terms a common denominator, we get:

$$10x+2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2. \qquad (1)$$

Method 1: (Comparing Coefficients) If we distribute terms, we get

$$0x^2 + 10x + 2 = (A+C)x^2 + (A+B-2C)x + (-2A+2B+C).$$

This gives the system of equations:

$$A+C=0 \qquad A+B-2C=10 \qquad -2A+2B+C=2$$

which we can solve for the values: $A=2$, $B=4$, $C=-2$.

Method 2: (Pluggin In Values) If we plug $x=1$ into (1) we get

$$10(1)+2 = A(1-1)(1+2) + B(1+2) + C(1-1)^2$$

This simplifies to give $12 = 3B$ and therefore $B=4$.
If we plug in $x=-2$

$$10(-2)+2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$$

This gives $-18 = 9C$ and therefore $C=-2$.
Then, plugging in B, C and $x=-1$, we get:

$$\begin{aligned} 10(-1)+2 &= A(-1-1)(-1+2) + 4(-1+2) + (-2)(-1-1)^2 \\ &\Rightarrow -8 = -2A + 4 - 8 \\ &\Rightarrow -4 = -2A \Rightarrow A = 2. \end{aligned}$$

So we have $A=2$, $B=4$, $C=-2$.

b. [6 points] Given the partial fraction decomposition

$$\frac{-x-10}{(x-3)(x^2+4)} = \frac{-1}{(x-3)} + \frac{x+2}{x^2+4},$$

evaluate the following indefinite integral, show all your steps:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx$$

Solution: Start by substituting and splitting up the integral:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = \int \frac{-1}{x-3}dx + \int \frac{x+2}{x^2+4}dx$$

Then we split up the second integral to get:

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = \int \frac{-1}{x-3}dx + \int \frac{x}{x^2+4}dx + \int \frac{2}{x^2+4}dx$$

For the first integral we have:

$$\int \frac{-1}{x-3}dx = -\ln|x-3| + C$$

For the second integral we use u-substitution with $u = x^2+4$ and $du = 2xdx$ to get

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(x^2+4) + C.$$

For the final integral, we first rewrite $\frac{2}{x^2+4} = \frac{2}{4((\frac{x}{2})^2+1)} = \left(\frac{1}{2}\right)\frac{1}{(\frac{x}{2})^2+1}$. Then we have:

$$\int \frac{2}{x^2+4}dx = \frac{1}{2}\int \frac{1}{(\frac{x}{2})^2+1}dx$$

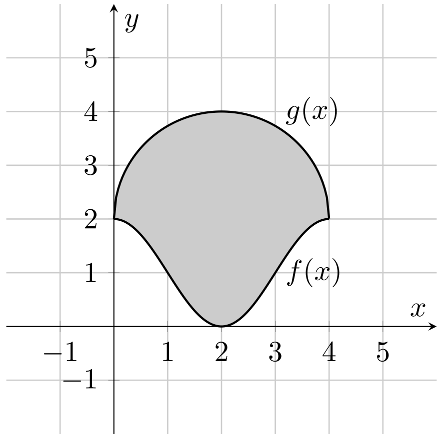
Using substitution with $u = \frac{x}{2}$, so $du = \frac{1}{2}dx$, this becomes:

$$\int \frac{2}{x^2+4}dx = \frac{1}{2}\int \frac{1}{(\frac{x}{2})^2+1}dx = \int \frac{1}{u^2+1}du = \arctan(u) + C = \arctan\left(\frac{x}{2}\right) + C$$

Putting this all together, we get

$$\int \frac{-x-10}{(x-3)(x^2+4)}dx = -\ln|x-3| + \frac{1}{2}\ln(x^2+4) + \arctan\left(\frac{x}{2}\right) + C.$$

4. [11 points] Consider the shaded region bounded by $f(x) = 2\cos^2\left(\frac{\pi x}{4}\right)$ and $g(x) = \sqrt{4-(x-2)^2}+2$ shown below.



a. [6 points] Write, but do not compute, an integral for the solid formed by rotating the region around the line $x = 5$.

Solution: Shell Method: The height of the shells are given by $h = g(x) - f(x)$, and the radius is the distance of the slice from the line $x = 5$. This is given by $r = 5 - x$. Since we are slicing along the x-axis for shell method, we integrate with respect to x from 0 to 4. Using the shell method formula, we get:

$$V = \int_0^4 2\pi(5-x)\left(\sqrt{4-(x-2)^2}+2-2\cos^2\left(\frac{\pi x}{4}\right)\right)dx$$

Washer Method: First, we note that this is significantly more complicated than Shell, as both $f(x)$ and $g(x)$ must be inverted to obtain functions of y. Both are also non-one-to-one making inversion more tedious. Secondly, we must split the integral at $y = 2$. Therefore we have the following. On $[0, 2]$:

$$r_{\text{in}}(y) = 5 - \frac{4}{\pi}\arccos\left(-\sqrt{\frac{y}{2}}\right) \qquad r_{\text{out}}(y) = 5 - \frac{4}{\pi}\arccos\left(\sqrt{\frac{y}{2}}\right)$$

This means the first component of the volume integral is:

$$V_1 = \int_0^2 \pi\left(\left(5-\frac{4}{\pi}\arccos\left(\sqrt{\frac{y}{2}}\right)\right)^2 - \left(5-\frac{4}{\pi}\arccos\left(-\sqrt{\frac{y}{2}}\right)\right)^2\right)dy$$

On $[2, 4]$, we have:

$$r_{in}(y) = 5 - \sqrt{4-(y-2)^2}+2 \qquad r_{out}(y) = 5 - \left(-\sqrt{4-(y-2)^2}\right)+2$$

giving the integral for this part as:

$$V_2 = \int_2^4 \pi\left(\left(5-\left(-\sqrt{4-(y-2)^2}\right)+2\right)^2 - \left(5-\left(\sqrt{4-(y-2)^2}\right)+2\right)^2\right)dy$$

The final answer is then $V_1 + V_2$, whose full form has been omitted for brevity.

b. [5 points] Write, but do not compute, an expression involving one or more integrals for the perimeter of the region above. *Hint: The upper curve is a semicircle.*

Solution: We find the area of the two curves separately and add them.

Lower curve: The arclength formula is required. First find $f'(x)$ which is given by

$$2\left(\frac{\pi}{4}\right)\left(\sin\left(\frac{\pi x}{4}\right)\right)\left(2\cos\left(\frac{\pi x}{4}\right)\right) = \pi\left(\sin\left(\frac{\pi x}{4}\right)\right)\left(\cos\left(\frac{\pi x}{4}\right)\right).$$

Then the arclength integral is given by:

$$L_1 = \int_0^4 \sqrt{1+\left(\pi\sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}dx$$

Upper curve with the hint: The hint says the upper curve is a semicircle. The circle has radius 2, and so the circumference of the (full) circle is given by 4π, with circumference of the semicircle given by 2π.

Using the hint, the final answer is

$$L = \int_0^4 \sqrt{1+\left(\pi\sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}dx + 2\pi$$

Upper curve without the hint: Use the arclength formula. First find $g'(x)$ as

$$g'(x) = \frac{(x-2)}{\sqrt{4-(x-2)^2}}.$$

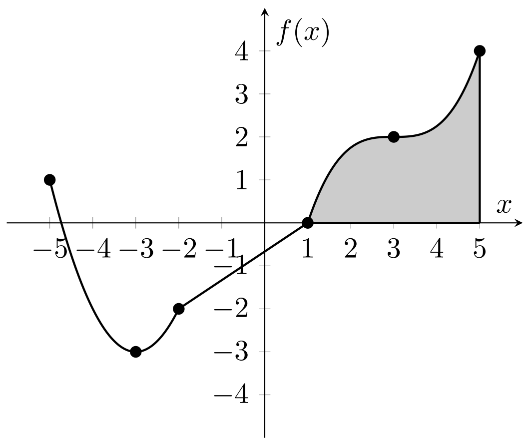
Then plug in the arclength formula to get

$$L_2 = \int_0^4 \sqrt{1+\left(\frac{(x-2)}{\sqrt{4-(x-2)^2}}\right)^2}dx$$

Without the hint, the final answer is:

$$L = \int_0^4 \sqrt{1+\left(\pi\sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)\right)^2}dx + \int_0^4 \sqrt{1+\left(\frac{(x-2)}{\sqrt{4-(x-2)^2}}\right)^2}dx$$

5. [13 points] The function below has a local minimum at $x = -3$, is linear on $[-2, 1]$, and has an inflection point at $x = 3$.



For parts **a.** and **b.**, use the graph of $f(x)$ to determine if the listed quantities are over- or under-estimates for the relevant integral, and write the word OVERESTIMATE or UNDERESTIMATE as appropriate. If there is not enough information, write NI.

a. [4 points] $\int_{-3}^{3} f(x)dx$

LEFT(4) **UNDER** RIGHT(4) **OVER**

MID(4) **N.I.** TRAP(4) **N.I.**

b. [4 points]
$\int_{-5}^{1} f(x)dx$

LEFT(12) **N.I.** RIGHT(12) **N.I.**

MID(12) **UNDER** TRAP(12) **OVER**

c. [5 points] The function $f(x)$ on $[1, 5]$ is given by the formula $\frac{1}{4}(x-3)^3 + 2$. Write, but do not solve, an integral giving the volume of the shaded region rotated around $y = -2$.

Solution: Washer Method: The inner radius is given by $r_{in}(x) = 2$ and the outer radius is by $r_{out}(x) = \frac{1}{4}(x-3)^3 + 4$.
Washer formula give:

$$\int_1^5 \pi \left(\left(\frac{1}{4}(x-3)^3 + 4 \right)^2 - 2^2 \right) dx$$

Shell Method: Invert the function so we get a function of y, $f(y) = (4(y-2))^{\frac{1}{3}} + 3$. Then our shells have height $5 - f(y)$ with radius $(2+y)$. Evalutating with the shell formula we have:

$$\int_0^4 2\pi(2+y) \left(5 - (4(y-2))^{\frac{1}{3}} + 3 \right) dy$$

6. [13 points]

a. [4 points] Which, if any, of the following are antiderivatives of the function e^{x^2}? Circle ALL that apply, or 'NONE' as appropriate.

$$\int_1^{x^2} \frac{e^t}{2t} dt \qquad \boxed{\int_{\sqrt{2}}^{x} e^{t^2} dt} \qquad \frac{e^{x^2}}{2x}$$

$$\int_0^{x^2} e^t dt \qquad \text{NONE}$$

b. [4 points] Back in his younger days, Brad hiked to the top of Mount Olympus. Let the continuous function $T(m)$ denote the rate of change of temperature, in degrees Celsius per meter, after Brad has hiked m meters. Suppose the following mathematical statements hold:

- $\displaystyle\int_0^{2500} T(m)dm = -25.$
- $\displaystyle -0.01 = \frac{1}{1000}\int_0^{1000} T(m)dm.$

Which of the following statements MUST be true? Circle ALL that apply, or 'NONE OF THE ABOVE' as appropriate.

i) $T(m)$ is negative for all values of m in its domain.

ii) The average rate of change of temperature per meter hiked was the same during the first 1000 meters Brad hiked as it was in the next 1500 meters he hiked.

iii) During the first 1000 minutes Brad was hiking, the temperature decreased by an average of 0.1 degrees Celsius per minute.

iv) The temperature decreased by 10 degrees Celsius during the first 1000 meters of Brad's hike.

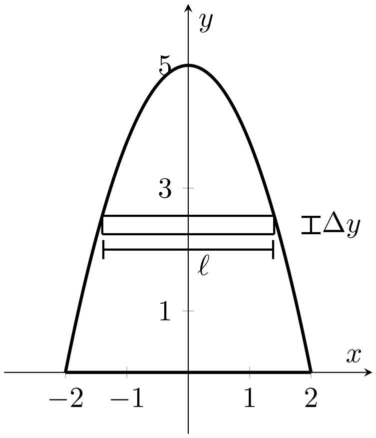
v) NONE OF THE ABOVE

c. [5 points] What is the mass of a solid cube with side length ℓ centimeters if its density x centimeters above its base is $x + 1$ grams per cubic centimeter for $0 \leq x \leq \ell$? Show all your work including evaluating all integrals and give your answer in terms of ℓ.

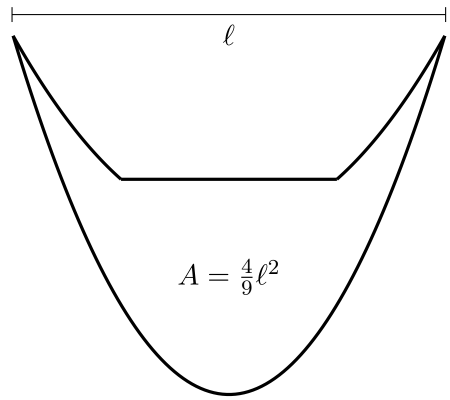
Solution: The integral is given by:

$$\int_0^{\ell} \ell^2(x+1)dx = \ell^2\left(\frac{x^2}{2} + x\right)\Bigg|_0^{\ell} = \frac{\ell^4}{2} + \ell^3$$

7. [13 points] Brad and Shawna are shipwrecked on an island and are building a new ship out of various materials. The ship has a base given by the region enclosed in the figure on the left, with cross-sections perpendicular to the y-axis given by the figure on the right. The base is the region bounded by $y = \frac{-5}{4}(x^2 - 4)$ and $y = 0$. The cross-sections have area given by $\frac{4}{9}\ell^2$ where ℓ is the length of the slice of the base directly below the cross-section. A sample slice of the base of thickness Δy is shown in graph on the left, and all distances are given in meters.



Base of Ship



Cross-section of Ship

a. [3 points] Write an expression for the length, ℓ, of a slice y meters from the x-axis. Give units.

Solution:

$$\ell = 2\sqrt{\frac{-4}{5}y + 4} \text{ m.}$$

b. [3 points] Write an expression for the volume of materials needed to construct a cross-sectional slice of the ship y meters from the x-axis with thickness Δy meters. The letter ℓ should not appear in your final answer. Give units.

Solution:

$$\frac{4}{9}\left(2\sqrt{\frac{-4}{5}y + 4}\right)^2 \Delta y \text{ m}^3 = \frac{16}{9}\left(\frac{-4}{5}y + 4\right)\Delta y \text{ m}^3$$

c. [3 points] The density of the materials used to make the ship varies. The materials used in the cross section y meters from the x-axis is given by $\delta(y) = (2y + 5)$ kg/m^3. What is the mass of a cross sectional slice y meters from the x-axis with thickness Δy meters? Give units.

Solution:

$$\frac{16}{9}(2y+5)\left(\frac{-4}{5}y + 4\right)\Delta y \text{ kg}$$

d. [4 points] Write an integral that gives the total mass of the new boat in kg. Do not evaluate your integral.

Solution:

$$\frac{16}{9}\int_0^5 (2y+5)\left(\frac{-4}{5}y+4\right)dy \text{ kg}$$

8. [12 points] After constructing their boat, Brad and Shawna departed the island, hoping to return home. However, their victory at Troy angered Poseidon, the god of the sea, who created large waves to further complicate their journey. The waves were so high Brad and Shawna could not see land, making navigation difficult. Hermes, the messenger of the gods, was sympathetic. He stole the formula Poseidon used to create the waves and gave it to Brad and Shawna. The formula is given by:

$$H(t) = 12\int_0^{\sin^2(t)} e^{x^2}\,dx.$$

The function $H(t)$ gives the wave height in meters, t minutes after Hermes stole the formula.

a. [7 points] Brad and Shawna can only see land at the moment their ship is at the top of a wave. If they know exactly when that time is coming, they can be prepared to correct their course toward land. If Hermes brought them the formula 4 minutes after he stole it, when is the first time they can see land? (Brad and Shawna did not know when to look for land before obtaining the formula.)

Solution: We find the first maximum of $H(t)$ after $t = 4$. We need to find where $H'(t) = 0$. By applying chain rule and 2^{nd} Fundamental Theorem of calculus, we find

$$H'(t) = 24\sin(t)\cos(t)e^{\sin^4(t)}$$

Since e^x is always positive, $e^{\sin^4(t)}$ can never be zero, so the critical points are where $\sin(t) = 0$ and $\cos(t) = 0$. These are $n\pi$ for $\sin(t)$ and $n\pi + \frac{\pi}{2}$ for $\cos(t)$. Now, we need to determine which are maximum and which are minimum. Since $\sin(n\pi) = 0$, if we plug these into $H(t)$, we see that $H(n\pi) = \int_0^0 e^{x^2}dx = 0$. If we check $n\pi + \frac{\pi}{2}$, $\sin(n\pi + \frac{\pi}{2}) = \pm 1$ so $\sin^2(n\pi + \frac{\pi}{2}) = 1$ and so $H(n\pi + \frac{\pi}{2}) = \int_0^1 e^{x^2}dx$, which is positive since e^{x^2} is always positive. Therefore, these are the maximum. If we start counting, we see that $\frac{\pi}{2} \approx 1.5$ and $\frac{3\pi}{2} \approx 4.5$ so the answer is $\frac{3\pi}{2}$.

b. [5 points] The same poet who recorded the tale of the Trojan war would like to record parts of Brad and Shawna's odyssey. Seeking a more appealing version of the expression above, he asks you for a different formula. Write a function equivalent to $H(t)$ with only t in the upper bound of the integral.

Solution: Using second fundamental theorem, we integrate $H'(t)$, which we solved for above. This gives

$$H(t) = \int_0^t 24\sin(x)\cos(x)e^{\sin^4(x)}dx$$