Math 116 — Second Midterm — March 21, 2022

Write your 8-digit UMID number very clearly in the box to the right.

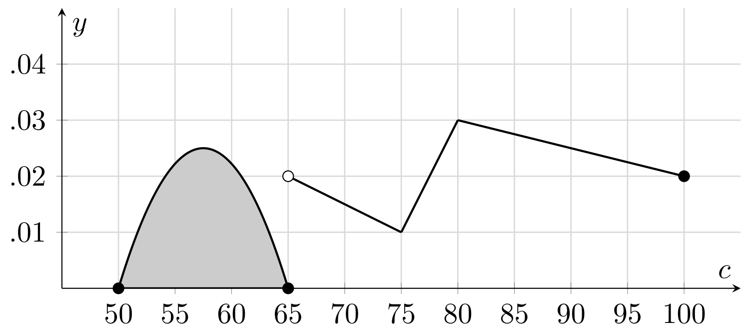
Your Initials Only: ______ Instructor Name: ____________________ Section #: ____

1. This exam has 14 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	7	
3	12	
4	15	
5	12	
6	12	

Problem	Points	Score
7	12	
8	8	
9	8	
10	8	
Total	100	

1. [6 points] Brad and Joan have developed a new strategy to analyze baseball players, except now instead of focusing on home run distance, they need to know the probability a pitcher throws a ball at a given speed. Shown below is a graph of the function $f(c)$, a probability density function (pdf) describing the probability a certain pitcher throws the ball at a speed of c miles per hour (mph). Assume that $f(c) = 0$ when $c \leq 50$ and $c > 100$.



a. [3 points] What is the probability this pitcher throws a pitch between 50 and 65 mph?

Solution: This probability is equal to $\int_{50}^{65} f(c)dc$, and so we are finding the area of the shaded region. Since the entire probability density is shown above, this area is equal to $1 - \int_{65}^{100} f(c)dc$. By breaking this area up into geometric shapes, we find that the area is 15 square units. Each unit has area equal to .05, so the total area is $15(.05) = .75$. Therefore, the final answer is $1 - .75 = .25$.

b. [3 points] What is the median speed of this player's pitches, in mph?

Solution: This can be done two ways. The first is finding M such that $\int_{50}^{M} f(c)dc = .5$. Using a), this is $.25 + \int_{65}^{M} f(c)dc = .5$, so this is equivalent finding $\int_{65}^{M} f(c)dc = .25$. Counting boxes shows that this happens at $c = 80mph$. The other way is to use the fact that $\int_{65}^{M} f(c)dc = .5$ is equivalent to $1 - \int_{M}^{100} f(c)dc = .5$. and so instead of counting boxes from left to right, we count boxes from right to left. This again gives the median as $80mph$.

2. [7 points] Brad and Joan are examining another pitcher's probability density function (pdf) when Brad spills coffee on the paper and smudges some of the ink. After drying off the paper, they are left with the incomplete probability density function, $g(v)$ given below, where v is in **hundreds of miles per hour.**

$$g(v) = \begin{cases} r + qv & 0 < v \leq 1 \\ 0 & \textit{otherwise} \end{cases}$$

Brad and Joan know that this player has a mean pitch speed of $\frac{2}{3}$ hundreds of miles per hour. Find the values of r and q which make this function a probability density function.

Solution: We need to find r and q such that the above function is a probability density function with the given mean. This means we need the equations

$$\int_0^1 r + qvdv = 1 \qquad \int_0^1 v(r+qv)dv = \frac{2}{3}$$

to be true. If we compute each integral, we get the equations

$$\int_0^1 r + qvdv = rv + \frac{qv^2}{2}|_0^1 = r + \frac{q}{2}$$

and

$$\int_0^1 v(r+qv)dv = \frac{rv^2}{2} + \frac{qv^3}{3}|_0^1 = \frac{r}{2} + \frac{q}{3}$$

so we just solve

$$r + \frac{q}{2} = 1 \qquad \frac{r}{2} + \frac{q}{3} = \frac{2}{3}$$

Solving this gives $r = 0$, $q = 2$

$r =$ **0** $\qquad q =$ **2**

3. [12 points]

a. [6 points] Suppose $F(x)$ is a cumulative distribution function for the height x, in meters, of the students on the University of Michigan campus. For each of the following, circle MUST BE, COULD BE, or CANNOT BE if the statement must be true, could be true, or cannot be true.

- $F(2) < F(1)$.
  MUST BE   COULD BE $\boxed{\text{CANNOT BE}}$

- $\lim\limits_{x\to\infty} F(x) = 1$.
  $\boxed{\text{MUST BE}}$ COULD BE   CANNOT BE

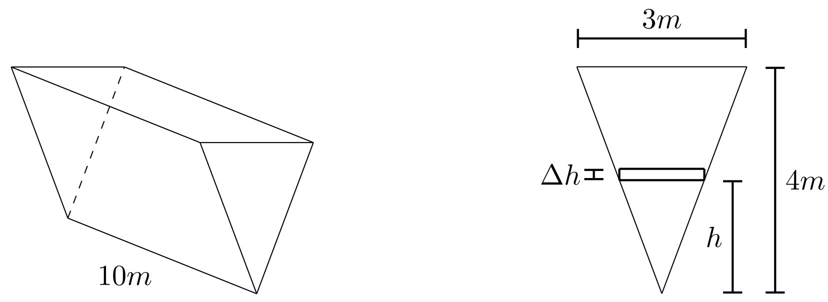
- $F(1.8) = 0.6$.
  MUST BE $\boxed{\text{COULD BE}}$ CANNOT BE

b. [6 points] Which of the following series converge? Circle all that apply. If none converge, circle NONE.

$$\sum_{n=1}^{\infty} \frac{e^n}{n} \qquad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.1}}} \qquad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}}$$

$$\sum_{n=1}^{\infty} e^{-1/n} \qquad \text{NONE}$$

4. [15 points] A gas station needs to pump gas out of a subterranean tank. The tank is 10 meters in length, and has cross-sections shaped like isosceles triangles, with base 3 meters and height 4 meters. **The top of the tank is 15 meters below the surface of the earth**. Recall that $g = 9.8m/s^2$ is the gravitational constant.



Underground Tank

a. [5 points] Write an expression for the volume (in cubic meters) of a horizontal rectangular slice of gasoline at height h above the bottom of the tank, with thickness Δh. Your answer should not involve an integral.

Solution: The slice has volume $\ell w \Delta h$. The length is constant at $10m$, so we just need to find the width as a function of height. Using similar triangles, $\frac{w}{h} = \frac{3}{4}$ so $w = \frac{3}{4}h$. Therefore the slice volume is $\frac{15}{2}h\Delta h\ m^3$.

b. [3 points] Gasoline has a density of 800 kg/m^3. Write an expression for the weight (in newtons) of the slice of gasoline mentioned in part (a). Your answer should not involve an integral.

Solution: Weight is the force exerted on a mass due to gravity, and so weight is mg. We compute the mass using the density from the problem statement and the volume from a). This means $m = (800)(\frac{15}{2}h)\Delta h kg$. Then the weight is

$$(9.8)(800)\left(\frac{15}{2}h\right)\Delta h\ N$$

.

c. [4 points] Write an expression for the work (in joules) needed to pump the slice of gasoline mentioned above to the surface of the earth. Your answer should not involve an integral.

Solution: Work is force times distance traveled. If the slice is h meters from the bottom of the tank, then the slice travels $(4-h)$ meters to get to the top of the tank. Then it travels the 15 meters to get from the top of the tank to the ground. Therefore, the total distance traveled is $19-h$ meters. The force is the weight from b), so the work for a slice is:

$$W_{\text{slice}} = (19-h)(9.8)(800)\left(\frac{15}{2}h\right)\Delta h$$

d. [3 points] Write an integral for the total work (in joules) needed pump all of the gasoline to the surface of the earth.

Solution: We integrate our work slices to find the total work. Since the slices range from $h=0$ to $h=4$, these are the bounds, so the work is given by the integral

$$\int_0^4 (19-h)(9.8)(800)\left(\frac{15}{2}h\right)\, dh$$

5. [12 points] A tech startup is growing quickly, and the company needs to understand its customers data-storage needs to properly scale its infrastructure. Over the course of each month, the users each store 5 gigabytes of new data. Additionally, because users are conscious of their digital footprint, at the beginning of each month, each user deletes 20% of all data they had stored in previous months.

a. [4 points] Let D_n be the amount of data stored per user at the end of the n^{th} month. If $D_1 = 5$, write expressions for D_2 and D_3. The letter D should not appear in your final answers.

$D_2 =$ $5 + 5\,(.8)$

$D_3 =$ $5 + 5\,(.8) + 5\,(.8)^2$

b. [4 points] Find a closed form expression for D_n. This means your answer should be a function of n, should not contain Σ, and should not be recursive.

$D_n =$ $\dfrac{5(1-(.8)^n)}{1-(.8)}$

c. [4 points] What is the long-term expected data storage of a user in gigabytes?

Answer = $\dfrac{5}{1-.8} = 25$

6. [12 points] Answer the following questions relating the the sequences shown here:

$$a_n = -\cos\left(\frac{\pi}{n}\right) \qquad b_n = \frac{(-1)^n(n+1)}{n} \qquad c_n = \left(\frac{4}{3}\right)^n \qquad d_n = \sum_{k=1}^{n}\left(-\frac{3}{4}\right)^k$$

Assume all sequences start at the index $n = 1$.

a. [3 points] Which of the sequences are bounded?

$\boxed{a_n}$ $\boxed{b_n}$ c_n $\boxed{d_n}$ *none*

b. [3 points] Which of the sequences shown above are monotone increasing?

a_n b_n $\boxed{c_n}$ d_n *none*

c. [3 points] Which of the sequences shown above are monotone decreasing?

$\boxed{a_n}$ b_n c_n d_n *none*

d. [3 points] Which of the sequences shown above converge?

$\boxed{a_n}$ b_n c_n $\boxed{d_n}$ *none*

7. [12 points] The parts of this problem are unrelated to each other.

a. [7 points] **Compute** the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. Be sure to show your full computation, and be sure to use **proper notation.**

$$\int_1^2 \frac{1}{\sqrt{t-1}}dt$$

Solution: First, this is an improper integral at $t = 1$. Therefore, we need to switch to limit notation:

$$\int_1^2 \frac{1}{\sqrt{t-1}}dt = \lim_{b \to 1^+} \int_b^2 \frac{1}{\sqrt{t-1}}dt$$

Now, we do a u-sub, with $u = t - 1$, so $du = dt$, so our integral becomes

$$\lim_{b \to 1^+} \int_{b-1}^1 \frac{1}{\sqrt{u}}du = \lim_{b \to 1^+} 2\sqrt{u}\Big|_{b-1}^1$$

Evaluating we get:

$$\lim_{b \to 1^+} (2\sqrt{1} - 2\sqrt{b-1}) = 2$$

b. [5 points] Compute the following limit. Fully justify your answer including using proper notation.

$$\lim_{x\to 0} \quad \frac{1-\cos(x)}{x^2}$$

Solution: As x goes to zero, this becomes an indeterminant form of $\frac{0}{0}$, so we apply L'Hopital to get

$$\lim_{x\to 0} \quad \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x}.$$

This is also an indeterminant form, so we use L'Hopital again, to get:

$$\lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2}.$$

Computing the final limit gives

$$\lim_{x\to 0} \frac{\cos(x)}{2} = \frac{\cos(0)}{2} = \frac{1}{2}.$$

So the final answer is

$$\lim_{x\to 0} \quad \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

8. [8 points] Determine whether the following improper integral converges or diverges. **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\int_1^\infty \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2} dt$$

Circle one: **Converges** **Diverges**

Solution: The numerator of the integrand is dominated by t^2, and the denominator is dominated by t^3, so this function has the same behavior as $\frac{t^2}{t^3} = \frac{1}{t}$, so we expect it to diverge. Therefore, we want to bound this function below by a function whose integral diverges. First, we note that $t^2 \le t^2 + \ln(t)$ on $[1, \infty)$. Then, for the denominator, since $\cos(x)$ oscillates from $[-1, 1]$, the denominator is largest(ans so the function is smallest) when $\cos(x) = -1$, so we get that $t^3 - \cos(t) + 2 \le t^3 + 1 + 2$, and so

$$\frac{t^2}{t^3+3} \le \frac{t^2}{t^3 - \cos(t) + 2} \le \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}$$

Next we know that $3 \le \frac{1}{2}t^3$ on $[2, \infty]$, and so $t^3 + 3 \le t^3 \frac{1}{2} t^3 = \left(\frac{3}{2}\right) t^3$, and so we get

$$\left(\frac{2}{3}\right)\frac{1}{t} = \left(\frac{2}{3}\right)\frac{t^2}{t^3} \le \frac{t^2}{t^3+3} \le \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}.$$

Then, $\dfrac{2}{3}\displaystyle\int_1^\infty \frac{1}{t} dt$ diverges by p-test, with $p = 1$, and so $\displaystyle\int_1^\infty \frac{t^2 + \ln(t)}{t^3 - \cos(t) + 2} dt$ diverges by comparison test, comparing $\left(\dfrac{2}{3}\right)\dfrac{1}{t} \le \dfrac{t^2 + \ln(t)}{t^3 - \cos(t) + 2}$

9. [8 points] Determine whether the following series converges or diverges. If it converges, determine if it is absolute or conditional convergence **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{n(-2)^n}{3^n}$$

Circle one: **Absolutely Converges** **Conditionally Converges** **Diverges**

Solution: We use the ratio test to show absolute convergence. Let $a_n = \dfrac{(-1)^n(n^{1/2}+4)}{3n^{3/2}+2}$. Then, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(-2)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{n(-2)^n}\right|$$

Since we take absolute value, we can drop our negative signs. Then, we can regroup terms and simplify :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{n}\cdot\frac{2^{n+1}}{2^n}\cdot\frac{3^n}{3^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{n}\cdot\frac{2}{1}\cdot\frac{1}{3}\right|$$

All of our terms are positive, so we can drop the absolute value signs. Therefore, we see that the limit is

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{(n+1)}{n} = \frac{2}{3}$$

where one can conclude $\displaystyle\lim_{n\to\infty}\frac{(n+1)}{n} = 1$ by L'Hopital, a dominating functions argument, or by writing $\frac{(n+1)}{n} = 1+\frac{1}{n}$. Since $\frac{2}{3} < 1$, this series converges absolutely by the ratio test.

10. [8 points] Determine whether the following series converges or diverges. If it converges, determine if it is absolute or conditional convergence **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n^{1/2}+4)}{3n^{3/2}+2}$$

Circle one: **Absolutely Converges** **Conditionally Converges** **Diverges**

Solution: First, we see that if we ignore the $(-1)^n$, checking leading terms tells us this series behaves like $\frac{1}{n}$, which should diverge. Since we do have the $(-1)^n$, this is an alternating series, and so we should be trying to show this is conditionally convergent.

To apply the alternating series test, we note that the sequence of $a_n = \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ is a decreasing sequence($a_n \geq a_{n+1}$), always positive ($a_n > 0$) , and the limit approaches zero $\lim\limits_{n\to\infty} a_n = 0$. Since the series is alternating, we satisfy the hypothesis of the alternating series test, and so the sequence converges.

To conclude conditionally convergent, we need to show the series:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n^{1/2}+4)}{3n^{3/2}+2} \right| = \sum_{n=1}^{\infty} \frac{(n^{1/2}+4)}{3n^{3/2}+2}$$

diverges. Like we noted above, this functions behaves like $\frac{1}{n}$, so we need to use limit comparison or comparison test with this series to show it diverges.

Solution: Limit Comparison: Let $a_n = \dfrac{(n^{1/2}+4)}{3n^{3/2}+2}$ and $b_n = \frac{1}{n}$. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{(n^{1/2}+4)}{3n^{3/2}+2}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n(n^{1/2}+4)}{3n^{3/2}+2} = \lim_{n\to\infty} \frac{n^{3/2}+4n}{3n^{3/2}+2}.$$

Using L'Hopital, or a dominating functions argument, we get:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}+4n}{3n^{3/2}+2} = \frac{1}{3}$$

Since this limit exists and is non-zero, the limit comparison test tells us that both $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{1/2}+4)}{3n^{3/2}+2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ will either both converge or both diverge. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test with $p = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{1/2}+4)}{3n^{3/2}+2}$ diverges by limit comparison test.

Solution: (Direct) Comparison Test: We can also use the direct comparison test. Since we are trying to conclude divergence, we must bound below by a divergent series.We need that $n^{1/2} \leq n^{1/2} + 2$ for all $n \geq 1$ to bound our numerator below. Then we use $3n^{3/2} + 2 \leq 3n^{3/2} + n^{3/2} = 4n^{3/2}$, and so

$$\frac{1}{4n^{3/2}} \leq \frac{1}{3n^{3/2} + 2}$$

for $n \geq 2$, to bound the denominator below. Putting this together, we get

$$\frac{1}{4n} = \frac{n^{1/2}}{4n^{3/2}} \leq \frac{(n^{1/2} + 4)}{3n^{3/2} + 2}.$$

Then, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test, with $p = 1$. Then, by (direct) comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{(n^{1/2} + 4)}{3n^{3/2} + 2}$ diverges.