Math 116 — Final Exam — April 22, 2022

Write your 8-digit UMID number very clearly in the box to the right.

Your Initials Only: ______ Instructor Name: ______________________ Section #: ____

1. This exam has 15 pages including this cover.
2. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	11	
3	16	
4	8	
5	16	
6	9	

Problem	Points	Score
7	6	
8	11	
9	10	
Total	100	

1. [13 points] Universe of Movies (UofM) is a new online movie rating database, which assigns movies a rating from zero to five stars. The star ratings for each movie are not necessarily integer values. The following **probability density function** gives the distribution of star rating, t, for the films in the UofM database.

$$p(t) = \begin{cases} 0 & t \le 0 \\ \frac{t}{6} & 0 < t \le 3 \\ \frac{t}{32} & 3 < t \le 5 \\ 0 & 5 < t \end{cases}$$

a. [6 points] Write a formula for the corresponding cumulative distribution function, $F(t)$.

$$F(t) = \begin{cases} 0 & t \le 0 \\ \frac{t^2}{12} & 0 < t \le 3 \\ \frac{t^2}{64} + \frac{39}{64} & 3 < t \le 5 \\ 1 & 5 < t \end{cases}$$

b. [3 points] What is the median number of stars for the films in the online database? Be sure to show any work.

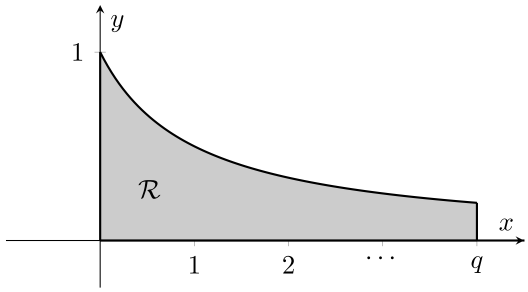
Solution: From finding the CDF in a), we know that $\int_0^3 p(t)dt = .75$ so the median is in $[0, 3]$. Therefore, we just need to solve $\int_0^M \frac{t}{6}dt = \frac{M^2}{12} = .5$ which when solved tells us that $M = \sqrt{6}$

Answer: $M = \sqrt{6}$

c. [4 points] Write, but do not compute, the formula for the mean number of stars for the movies in the UofM database. Write out formulas of all functions you use, i.e. do not include p or F in your answers.

Answer: $\displaystyle\int_0^3 \frac{t^2}{6}dt + \int_3^5 \frac{t^2}{32}dt$

2. [11 points] Consider the function $f(x) = e^{-2x}$, and the region $\mathcal{R}$ bounded by the x-axis, the y-axis, $y = f(x)$ and $x = q$, where q is a positive constant larger than 2.



a. [4 points] Give a formula for, but do not compute, the volume of the solid formed by rotating the region $\mathcal{R}$ around the y-axis. Your answer should depend on q. *(Hint: Use the shell method)*

Solution: Using shell method, we see that $V = \int_0^q 2\pi x e^{-2x} dx$.
Using washer method, we get:

$$\int_0^{e^{-2q}} \pi q^2 dy + \int_{e^{-2q}}^{1} \pi \left(\frac{1}{2}\ln(y)\right)^2 dy$$

b. [4 points] Compute the integral you found in part a). Your final answer should be in terms of q.

Solution: Using shell method from a) allows us use integration by parts with $u = x$ and $dv = e^{-2x}dx$ to give $du = dx$ and $v = \frac{-1}{2}e^{-2x}$. Putting this together gives:

$$\int_0^q 2\pi x e^{-2x} dx = -\pi x e^{-2x}\Big|_0^q + \pi \int_0^q e^{-2x} dx = \pi \left[-xe^{-2x} - \frac{1}{2}e^{-2x}\right]\Big|_0^q$$

When we evaluate FTC, we get:

$$V = \pi \left[-qe^{-2q} - \frac{1}{2}e^{-2q} - 0 + \frac{1}{2}\right]$$

c. [3 points] Taking a limit of your answer in *b*), compute the volume of the infinitely long solid of revolution formed by rotating the region $\mathcal{R}$ around the y-axis. Be sure to show how you got the value of your limit.

Solution: An infinitely long solid means $q \to \infty$. Therefore we get

$$V = \lim_{q\to\infty} \pi \left[qe^{-2q} - \frac{1}{2}e^{-2q} - 0 + \frac{1}{2} \right] = \pi[0 - 0 + 0 + \frac{1}{2}] = \frac{\pi}{2}.$$

where the first limit vanishes by L'Hopital or dominating functions argument, and the second is just a standard limit.

3. [16 points] Use the table below to answer the following questions. Write your answers using **exact form** on the blank provided. If there is not enough information to complete the problem, write N.I. You need to evaluate all integrals completely, but you do not need to simplify your final answers.

t	0	1	2	3	4	5
$f(t)$	3	2	5	e	7	0
$f'(t)$	-1	-4	3	2	π	1
$f''(t)$	2	8	1	2	6	2

a. [4 points] MID(2) for $\int_1^5 f(t)dt$

Answer: 24.

Solution: $\Delta t = 2$ which means the intervals are $[1,3]$ and $[3,5]$ with midpoints then being 2 and 4 respectively.

$$f(t_1) \cdot \Delta t + f(t_2) \cdot \Delta t = f(2) \cdot 2 + f(4) \cdot 2 = 24$$

b. [4 points] TRAP (2) for $\int_0^4 tf'(t)dt$

Answer: $4\pi + 12$.

Solution: TRAP(2) = (LEFT(2) + RIGHT(2))/2, $\Delta t = 2$, so we have

$$2(0f'(0) + 2 \cdot 2f'(2) + 4f'(4))/2 = 0 + 12 + 4\pi$$

c. [4 points] Average value of $f'(2t)$ on $[0, 2]$

Answer: **1**.

Solution: Average value$= \frac{\int_0^2 f'(2t)dt}{(2-0)} = \frac{1}{4}(f(4) - f(0)) = 1$

d. [4 points] Second degree Taylor Polynomial of $f(-x)$ around $x = -2$.

Answer: $5 - 3(x+2) + \frac{1}{2}(x+2)^2$.

Solution: Note that $f(-x)$ at $x = -2$ is $f(2)$, the first derivative is $-f'(2)$ and the second derivative is $f''(2)$. Then, since we are around $x = -2$, we need out terms to be of the form $(x+2)$. Putting this all together in the formula for second degree gives us

$$f(2) - f'(2)(x+2) + \frac{f''(2)}{2}(x+2) = 5 - 3(x+2) + \frac{1}{2}(x+2)^2$$

4. [8 points] Suppose a_n and b_n are sequences of positive numbers, defined for $n = 1, 2, 3 \ldots$, satisfying the following conditions:

- $a_n \le \frac{1}{n^{1/2}}$
- $b_n \ge \frac{1}{n}$

For each statement below, circle ALWAYS if the statement is always true, SOMETIMES if the statement can be true or false depending on the specifics of a_n or b_n, and NEVER if the statement is false for all specific a_n or b_n.

a. [1 point] $\lim_{n\to\infty} a_n = 0$.

$\boxed{\text{ALWAYS}}$ SOMETIMES NEVER

b. [1 point] $\lim_{n\to\infty} b_n = 0$.

ALWAYS $\boxed{\text{SOMETIMES}}$ NEVER

c. [1 point] a_n is bounded.

$\boxed{\text{ALWAYS}}$ SOMETIMES NEVER

d. [1 point] b_n is monotone.

ALWAYS $\boxed{\text{SOMETIMES}}$ NEVER

e. [1 point] $\sum_{n=1}^{\infty} a_n$ converges.

ALWAYS $\boxed{\text{SOMETIMES}}$ NEVER

f. [1 point] $\sum_{n=1}^{\infty} b_n$ converges.

ALWAYS SOMETIMES $\boxed{\text{NEVER}}$

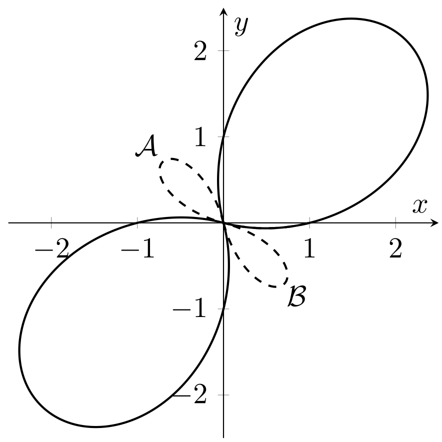
g. [1 point] $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

ALWAYS $\boxed{\text{SOMETIMES}}$ NEVER

h. [1 point] $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

ALWAYS $\boxed{\text{SOMETIMES}}$ NEVER

5. [16 points] The following problems relate to the polar graph shown below, defined by the polar curve $r(\theta) = 2\sin(2\theta) + 1$, on the domain $[0, 2\pi]$. Both the dashed and solid curves are part of the graph of $r(\theta)$.



a. [4 points] Find all θ values in the interval $[0, 2\pi]$ such that $r(\theta) = 0$.

Solution: We set $r(\theta) = 2\sin(2\theta) + 1 = 0$ and solve. This becomes:

$$\sin(2\theta) = \frac{-1}{2}$$

This means $2\theta = \frac{-\pi}{6} + 2k\pi$ or $\frac{7\pi}{6} + 2k\pi$ Now if we divide by 2, we get $\theta = \frac{-\pi}{12} + k\pi$ or $\frac{7\pi}{12} + k\pi$. Now, since we need to be in the interval$[0, 2]$, we take $k = 1, 2$ for the first possible term to get:

$$\theta = \frac{-\pi}{12} + \pi = \frac{11\pi}{12} \qquad \theta = \frac{-\pi}{12} + 2\pi = \frac{23\pi}{12}.$$

We take $k = 0, 1$ for the second possible term to get:

$$\theta = \frac{7\pi}{12} + 0\pi = \frac{7\pi}{12} \qquad \theta = \frac{7\pi}{12} + \pi = \frac{19\pi}{12}$$

as answers

Answers: $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b. [4 points] Determine the θ intervals corresponding to the dashed portions $\mathcal{A}$ and $\mathcal{B}$ of the curve above.

Solution: We need to examine when our radius is changing sign. First, we see that at $\theta = 0$, the radius is positive. It must stay positive, sweeping out the large undotted portion in the first quadrant, until $\theta = \frac{7\pi}{12}$, which is when the radius is first zero, based on our work in a). At this point, the radius becomes negative. The next zero occurs at $\frac{11\pi}{12}$, which means that for the interval $[\frac{7\pi}{12}, \frac{11\pi}{12}]$, the radius is negative, which should sweep out the dotted curve $\mathcal{B}$. Continuing in this way, we see that the next interval $[\frac{11\pi}{12}, \frac{19\pi}{12}]$ will correspond to the next solid region, and then the interval $[\frac{19\pi}{12}, \frac{23\pi}{12}]$, we trace out the curve $\mathcal{A}$.

Interval for $\mathcal{A}$: $[\frac{19\pi}{12}, \frac{23\pi}{12}]$ Interval for $\mathcal{B}$: $[\frac{7\pi}{12}, \frac{11\pi}{12}]$

c. [4 points] Write an expression involving one or more integrals for the area of the region enclosed by the **solid** curves only (do not include the region enclosed by the dashed curves).

Solution: The solid portions are where the dashed portions are not. Therefore they are defined by the angles which are not our answers in b). Since we still need to be in $[0, 2\pi]$, This means our answers are:

$$\frac{1}{2}\int_0^{\frac{7\pi}{12}} \left(2\sin\left(2\theta\right)+1\right)^2 d\theta + \frac{1}{2}\int_{\frac{11\pi}{12}}^{\frac{19\pi}{12}} \left(2\sin\left(2\theta\right)+1\right)^2 d\theta + \frac{1}{2}\int_{\frac{23\pi}{12}}^{2\pi} \left(2\sin\left(2\theta\right)+1\right)^2 d\theta$$

d. [4 points] Write an expression involving one or more integrals for the total arc length of the **dashed** curves in the graph above.

Solution: We can do this two ways. The first, is to have one integral for each curve. Using the polar arclength formula we get:

$$\int_{\frac{7\pi}{12}}^{\frac{11\pi}{12}} \sqrt{\left(2\sin\left(\theta\right)+1\right)^2 + \left(4\cos(2\theta)\right)^2}\,d\theta + \int_{\frac{19\pi}{12}}^{\frac{23\pi}{12}} \sqrt{\left(2\sin\left(\theta\right)+1\right)^2 + \left(4\cos(2\theta)\right)^2}\,d\theta$$

The second is to double one integral. Using the polar arclength formula we get:

$$2\int_{\frac{7\pi}{12}}^{\frac{11\pi}{12}} \sqrt{\left(2\sin\left(\theta\right)+1\right)^2 + \left(4\cos(2\theta)\right)^2}\,d\theta$$

6. [9 points] Suppose the Taylor series for a function $f(x)$ around $x = 3$ is

$$\sum_{n=1}^{\infty} \frac{(6)^{-n}((2n)!)}{n!(n-1)!}(x-3)^{2n}$$

a. [6 points] Compute the radius of convergence for this series. Be sure to fully justify your answer and show all work. Do not compute the interval of convergence.

Solution: We use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(6)^{-(n+1)}((2n+2)!)}{(n+1)!n!}\frac{n!(n-1)!}{(6)^{-(n)}((2n)!)}\frac{(x-3)^{2n+2}}{(x-3)^{2n}}\right| < 1$$

Pairing terms to cancel we get:

$$\lim_{n\to\infty}\left|\frac{(6)^{-(n+1)}}{(6)^{-(n)}}\frac{((2n+2)!)}{((2n)!)}\frac{n!(n-1)!}{(n+1)!n!}(x-3)^2\right| < 1$$

Then simplifying we are left with:

$$\lim_{n\to\infty}\left|\frac{1}{6}\cdot\frac{(2n+1)(2n+1)}{(n+1)n}\cdot(x-3)^2\right| < 1$$

Then, if we take the limit, by L'Hopital or a dominating functions argument, the radius, we get:

$$\frac{2}{3}\left|x-3\right|^2 < 1$$

which means $|x-3| < \left(\frac{3}{2}\right)^{1/2}$.

Radius of Convergence: $\sqrt{\frac{3}{2}}$

b. [3 points] Find $f^{(2022)}(3)$ and $f^{(2023)}(3)$.

Solution: A power series is its own Taylor series, so we get we just need to find n such that :

$$\frac{f^{(2022)}(3)}{(2022!)}(x-3)^{2022} = \frac{(6)^{-n}((2n)!)}{n!(n-1)!}(x-3)^{2n}$$

Comparing powers of $(x-3)$, we see that $2n = 2022$, and so $n = 1011$. This means:

$$\frac{f^{(2022)}(3)}{(2022!)} = \frac{(6)^{-1011}((2022)!)}{(1011!)(1010!)}$$

and so

$$f^{(2022)}(3) = \frac{(6)^{-1011}(2022!)^2}{(1011!)(1010!)}$$

Since the series is even, $f^{(2023)}(3) = 0$

$f^{(2022)}(3) =$ $\frac{(6)^{-1011}(2022!)^2}{(1011!)(1010!)}$ $\qquad$ $f^{(2023)}(3) =$ 0

7. [6 points] Compute the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(x-4)^n$$

Show all work and give all necessary justification. You may assume it has radius of convergence 4 and you do not need to show this.

Solution: We have that the radius of convergence is 4. We know that the series is centered at $x = 4$. Therefore, to compute the interval of convergence, we just need to check convergence at the endpoints, which occur at 0 and 8. At $x = 0$, the series becomes:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(0-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

This diverges by p-test $p = 1$, so the series does not converge at $x = 0$, meaning the interval is now $(0, 8?$.
If we plug in $x = 8$, we get:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n4^n}(8-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n(4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

As the terms $a_n = \frac{1}{n}$ are decreasing and going to zero, this series converges by the alternating series test. Since the series converges at $x = 8$, we include it in the radius of convergence

Interval of Convergence: $(0, 8]$

8. [11 points]

a. [6 points] Give the first three non-zero terms of the Taylor Series for the function:

$$(x^2+2)\sin(x)$$

centered at $x = 0$.

Solution: The function (x^2+2) is a polynomial, and therefore its own Taylor series. Next, we take the known Taylor series for $\sin(x)$. Since we need the first three non-zero terms, we start with the first 3 non-zero terms of $\sin(x)$. This gives:

$$(x^2+2)\sin(x) \approx (x^2+2)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5\right)$$

If we expand the right side, we get:

$$x^3 - \frac{1}{3!}x^5 + \frac{1}{5!}x^7 + 2x - \frac{2}{3!}x^3 + \frac{2}{5!}x^5$$

Now, if we group terms, we get:

$$2x + (1 - \frac{2}{3!})x^3 + (\frac{2}{5!} - \frac{2}{3!})x^5 + \frac{1}{5!}x^7$$

Nothing has cancelled out, meaning the first three terms here are the first three non-zero terms we need. Therefore the final answer is:

$$2x + \left(1 - \frac{2}{3!}\right)x^3 + \left(\frac{2}{5!} - \frac{2}{3!}\right)x^5$$

b. [5 points] Compute the limit:

$$\lim_{x\to 0} \frac{\int_0^{2x} \left(\left(\frac{t}{2}\right)^2 + 2\right)\sin\left(\frac{t}{2}\right)dt}{x^2}$$

Solution: Solutions 1 and 2): If we examine the bounds of the integral, we see that as $x \to 0$, the numerator becomes zero. This means that the answer is in indeterminant form, and we can apply L'Hopital. Using second FTC, and chain rule, we get that the limit becomes:

$$\lim_{x\to 0} \frac{2((x^2+2)\sin(x))}{2x} = \lim_{x\to 0} \frac{(x^2+2)\sin(x)}{x}$$

From here, the limit becomes $\frac{0}{0}$ again, so we have two options. We can apply L'Hopital again, turning the limit into:

$$\lim_{x\to 0} \frac{(2x\sin(x) + (x^2+2)\cos(x))}{1} = \frac{(0+(0^2+2)\cos(0))}{1} = 2$$

Or, instead of L'Hopital, we can plug in the answer from a) to get:

$$\lim_{x\to 0} \frac{2x + \left(1-\frac{2}{3!}\right)x^3 + \left(\frac{2}{5!}-\frac{2}{3!}\right)x^5}{x} = \lim_{x\to 0} 2 + \left(1-\frac{2}{3!}\right)x^2 + \left(\frac{2}{5!}-\frac{2}{3!}\right)x^4 = 2+0+0 = 2.$$

Solution 3): It is also possible to integrate the answer from a). Doing a u-sub, with $u = t/2$, before taking the limit, we get

$$\lim_{x\to\infty} \frac{2\int_0^x (u^2+2)\sin(u)du}{x^2}$$

Subbing in the answer from a) we get:

$$\lim_{x\to\infty} \frac{2\int_0^x 2u + \left(1-\frac{2}{3!}\right)u^3 + \left(\frac{2}{5!}-\frac{2}{3!}\right)u^5)du}{x^2}$$

This becomes:

$$\lim_{x\to\infty} \frac{2(x^2 + \frac{1}{4}\left(1-\frac{2}{3!}\right)x^4 + \frac{1}{6}\left(\frac{2}{5!}-\frac{2}{3!}\right)x^6)}{x^2}$$

Which becomes:

$$\lim_{x\to\infty} 2(1 + \frac{1}{4}\left(1-\frac{2}{3!}\right)x^2 + \frac{1}{6}\left(\frac{2}{5!}-\frac{2}{3!}\right)x^4) = 2(1+0+0) = 2$$

Answer: 2

9. [10 points] After a mistake on their last mission, Brad and Angelina must go on the run together, or risk being captured by an opposing agent. Brad and Angelina's shared position is given by the parametric equations:

$$(f(t), g(t)) = (t^2 + 10,\ 2t^2 + 10)$$

and the agent pursuing them has position given by the equations

$$(r(t), q(t)) = (7t,\ \sin(\pi t) + 12t).$$

The time t is measured in hours after Brad and Angelina have gone on the run, and all distances are given in miles.

a. [5 points] The agent catches up with Brad and Angelina at the smallest positive t-value when the agent is in the same position as they are. Find the time when the agent catches up with Brad and Angelina.

Solution: Start by comparing the x coordinates. To be in the same place at the same time, we need $t^2 + 10 = 7t$. This is the same as solving $t^2 - 7t + 10 = 0$. This polynomial factors as $(t-2)(t-5) = 0$. So the only times the x coordinates are the same is at $t = 2, 5$. Now, we just check that these work for the y-coordinates. If we plug in $t = 2$, $g(2) = 8 + 10$, where $q(2) = 0 + 24$, since $18 \neq 24$, the times $t = 2$ fails. However, if we plug in $t = 5$, we see that $g(5) = 50 + 10 = 60$ and $q(5) = 0 + 60 = 60$. So the time $t = 5$ is a solution, and therefore must be the only solution.

Answer: $t = 5$

b. [5 points] Compute the total distance traveled by Brad and Angelina before the agent catches up with them.

Solution: We will need to use the parametric arclength formula. Since the agent catches them at $t = 5$, this is the upper bound, and $t = 0$ is the lower bound. $f'(t) = 2t$, and $g'(t) = 4t$. Therefore, the arclength interal becomes:

$$\int_0^5 \sqrt{(2t)^2 + (4t^2)}dt = \int_0^5 \sqrt{4t^2 + 16t^2}dt = \int_0^5 \sqrt{20}tdt.$$

Integrating gives:

$$\int_0^5 \sqrt{20}tdt = \frac{\sqrt{20}}{2}t^2\Big|_0^5 = 25\sqrt{5}\text{miles}.$$

Answer: $25\sqrt{5}$**miles**

"Known" Taylor series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \leq 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin\theta$	$\cos\theta$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$