Math 116 — First Midterm — February 6, 2023

Write your 8-digit UMID number very clearly in the box to the right.

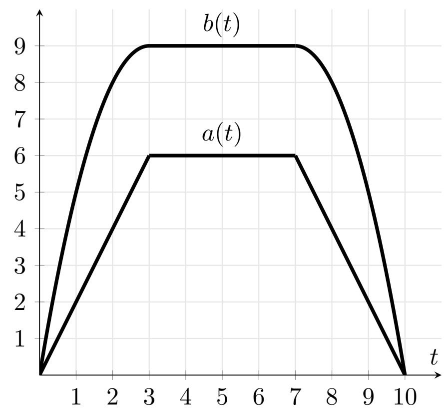
Your Initials Only: ______ Instructor Name: ____________________ Section #: ____

1. This exam has 13 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
7. Include units in your answer where that is appropriate.
8. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	18	
3	12	
4	15	
5	8	
6	16	

Problem	Points	Score
7	10	
8	12	
Total	100	

1. [9 points] Minhea, a production manager at ChipCorp, is comparing the performances of two machines that produce cellphone chipsets. One machine is new, and the other is old. Both machines are switched on at the same time each day. The **rate** of chipset production for the old machine and the new machine, in **hundreds** of chipsets per hour, are given by the functions $a(t)$ and $b(t)$ respectively, where t is the number of hours elapsed since the machines are switched on. The machines run for exactly 10 hours each day. The graphs of $a(t)$ and $b(t)$ are provided below.



The functions a and b have the following properties:

- $\int_0^3 b(t)\,dt = \int_7^{10} b(t)\,dt = 18.$
- $a(t)$ is piecewise linear on $0 \le t \le 10$.
- $b(t)$ is linear on $3 \le t \le 7$.

a. [4 points] Four hours after **both** the old and the new machines were switched on, a ChipCorp worker reports to Minhea that ChipCorp currently has 17,600 chipsets. What is the number of chipsets ChipCorp had in inventory before the machines were switched on? You need not simplify your answer, but your final answer must not involve any integrals.

Solution: Let x be the number (in hundreds) of chipsets ChipCorp had in inventory before the machines were switched on. Then, by the first FTC, we know that

$$x + \int_0^4 a(t)\,dt + \int_0^4 b(t)\,dt = 176.$$

Solving, we have,

$$\begin{aligned} x + \int_0^4 a(t)\,dt + \int_0^4 b(t)\,dt &= 176 \\ x + \int_0^3 a(t)\,dt + \int_3^4 a(t)\,dt + \int_0^3 b(t)\,dt + \int_3^4 b(t)\,dt &= 176 \\ x + 9 + 6 + 18 + 9 &= 176 \\ x &= 134. \end{aligned}$$

Answer: **13400 chipsets**.

b. [5 points] In her comparative study, Minhea is calculating the time it takes for the new machine to produce as many chipsets as the old machine would in an entire day. Find the time, T, for which the total chipsets produced by the new machine in its first T hours of operation equals the total chipsets produced by the old machine in its full day (i.e. 10 hours) of operation. If there is no such time, T, write "none", and explain why.

Solution: We solve the following equation for T:

$$\int_0^T b(t)\,dt = \int_0^{10} a(t)\,dt.$$

First, we compute, from the graph, that $\int_0^{10} a(t)\,dt = 42$. Next, we notice that $\int_0^3 b(t)\,dt = 18$ (which is less than 42). So, if a T exists, it has to be greater than 3. We rewrite our equation as,

$$\begin{aligned}\int_0^3 b(t)\,dt + \int_3^T b(t)\,dt &= 42\\ \int_3^T b(t)\,dt &= 24.\end{aligned}$$

Now, we note that $\int_3^7 b(t)\,dt = 36$ (which is larger than 24). Therefore, we must have $3 < T < 7$. Knowing this, we have that

$$\begin{aligned}\int_3^T b(t)\,dt &= 24\\ 9\cdot(T-3) &= 24\\ T &= 3 + \frac{8}{3} = \frac{17}{3}.\end{aligned}$$

Answer: $T =$ $\mathbf{\frac{17}{3}}$.

2. [18 points] The table below provides some values for the functions h and H, where

- $h(t)$ is an **odd** function, with continuous first derivative.
- $H(t)$ is an antiderivative of $h(t)$.

t	1	2	3	4	5
$h(t)$	-8	1	-2	4	$\sqrt{\pi}$
$H(t)$	-3	0	-5	3	6

Use the table above to compute the following integrals. Write your answers using **exact form** on the blank provided. If there is not enough information to answer a question, write "N.I." Evaluate all integrals. You do not need to simplify your answers, but the letters h or H should not appear in your final answers.

a. [4 points] $\displaystyle\int_{-4}^{4} h'(t)\,dt$

Solution: By the first FTC, and the fact that $h(-4) = -h(4) = -4$ (since h is an odd function), we have

$$\int_{-4}^{4} h'(t)\,dt = h(4) - h(-4) = 4 - (-4) = 8.$$

Alternatively, we can note that if h is an odd function (i.e. it satisfies $h(t) = -h(-t)$), then h' is an even function (as it satisfies $h'(t) = -h'(-t) \cdot -1 = h'(-t)$). Therefore,

$$\int_{-4}^{4} h'(t)\,dt = 2\int_{0}^{4} h'(t)\,dt = 2(h(4) - h(0)) = 2(4 - 0) = 8,$$

where we have used the fact that $h(0) = 0$ since h is an odd function.

Answer: **8**.

b. [4 points] $\displaystyle\int_{3}^{2} th'(t)\,dt$

Solution: We perform integration by parts, with

$$\begin{aligned} u &= t, & du &= dt, \\ dv &= h'(t)\,dt, & v &= h(t), \end{aligned}$$

to get,

$$\begin{aligned} \int_{3}^{2} th'(t)\,dt &= th(t)\Big|_{3}^{2} - \int_{3}^{2} h(t)\,dt \\ &= (2h(2) - 3h(3)) - (H(2) - H(3)) \\ &= (2 \cdot 1 - 3 \cdot (-2)) - (0 - (-5)) \\ &= 2 + 6 - 5 = 3. \end{aligned}$$

Answer: **3**.

c. [4 points] $\displaystyle\int_1^2 \frac{\cos((h(t))^{\frac{1}{3}})}{(h(t))^{\frac{2}{3}}} h'(t)\,dt$

Solution: Let $u = (h(t))^{\frac{1}{3}}$. This gives us $du = \frac{1}{3}\frac{1}{(h(t))^{\frac{2}{3}}} h'(t)\,dt$, and the change of bounds $t = 1 \to u = (h(1))^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = -2$, and $t = 2 \to u = (h(2))^{\frac{1}{3}} = 1^{\frac{1}{3}} = 1$. So, we get

$$\int_1^2 \frac{\cos((h(t))^{\frac{1}{3}})}{(h(t))^{\frac{2}{3}}} h'(t)\,dt = \int_{-2}^1 3\cos u\,du = 3\sin u\Big|_{-2}^1 = 3\sin 1 - 3\sin(-2) = 3\sin 1 + 3\sin 2.$$

Answer: $3\sin 1 - 3\sin(-2)$.

d. [6 points] $\displaystyle\int_2^5 \frac{h(t)}{1+(h(t))^4} h'(t)\,dt$

Solution: Let $u = (h(t))^2$. This gives us $du = 2h(t)h'(t)\,dt$, and the change of bounds $t = 2 \to u = (h(2))^2 = 1$, and $t = 5 \to u = (h(5))^2 = \pi$. So, we get

$$\int_2^5 \frac{h(t)}{1+(h(t))^4} h'(t)\,dt = \frac{1}{2}\int_1^\pi \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u\Big|_1^\pi = \frac{1}{2}(\arctan\pi - \arctan 1) = \frac{1}{2}\Big(\arctan\pi - \frac{\pi}{4}\Big).$$

Answer: $\frac{1}{2}\arctan\pi - \frac{\pi}{8}$.

3. [12 points] Let $f(x) = \dfrac{9-x}{(x+3)(x^2+3)}$.

a. [7 points] Split the function $f(x)$ into partial fractions with two or more terms. Do not integrate the result. Be sure to show all your work.

Solution: As we have a linear factor and an irreducible (unfactorable) quadratic in the denominator of $f(x)$, we seek a partial fraction decomposition of the form

$$\frac{9-x}{(x+3)(x^2+3)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+3}.$$

By giving terms on the right hand side a common denominator, we get the following equation for the numerators,

$$9 - x = A(x^2+3) + (Bx+C)(x+3).$$

If we distribute the coefficients, we get

$$9 - x = (A+B)x^2 + (3B+C)x + 3(A+C),$$

resulting in the following system of equations,

$$\begin{aligned} A + B &= 0, \\ 3B + C &= -1, \\ 3A + 3C &= 9, \end{aligned}$$

which we can solve to obtain $A = 1, B = -1$, and $C = 2$.

Answer: $f(x) = \dfrac{1}{x+3} + \dfrac{-x+2}{x^2+3}$.

b. [3 points] Approximate the integral $\displaystyle\int_{-9}^{-5} f(x)\,dx$ using MID(2). Write out each term in your sum. You do not need to simplify the numbers in your sum, but the letter f should not appear in your final answer.

Solution: As we are using MID(2), we divide the interval $[-9, -5]$ into the two equal sub-intervals $[-9, -7]$ and $[-7, -5]$. The midpoints of the sub-intervals are $x = -8$ and $x = -6$ respectively, whereas the width of each of the sub-intervals is 2. Therefore, we have

$$\text{MID}(2) = 2 \cdot \left(\frac{9-(-8)}{((-8)+3)((-8)^2+3)} + \frac{9-(-6)}{((-6)+3)((-6)^2+3)} \right)$$

Answer: $\displaystyle\int_{-9}^{-5} f(x)\,dx \approx 2 \cdot \left(\left(\frac{1}{(-8)+3} + \frac{-(-8)+2}{(-8)^2+3} \right) + \left(\frac{1}{(-6)+3} + \frac{-(-6)+2}{(-6)^2+3} \right) \right)$.

c. [2 points] Given that $f'(x)$ is decreasing on the interval $(-9,-5)$, is your answer to part **b.** an overestimate or an underestimate of $\int_{-9}^{-5} f(x)\,dx$? Circle your choice below. You are not required to provide any justification.

Circle one:

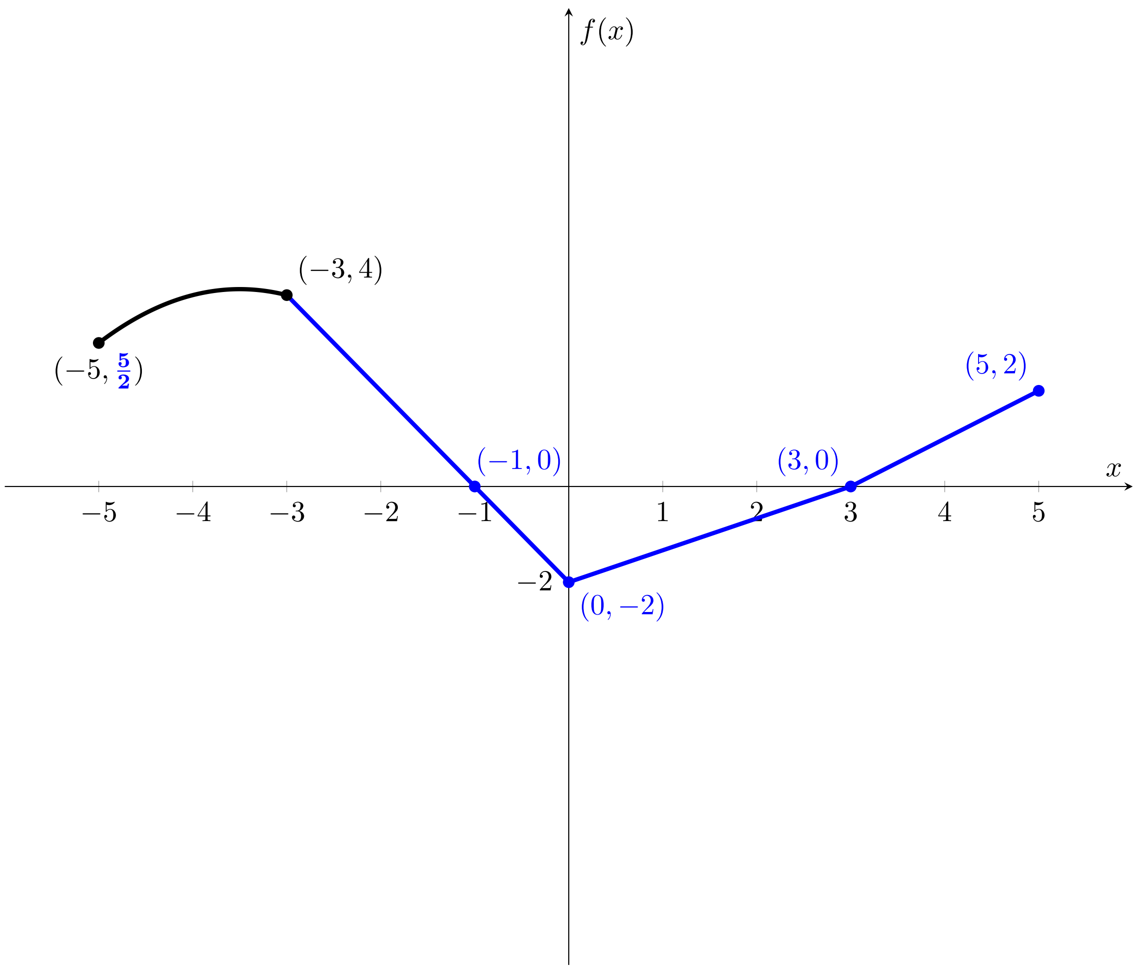
$\boxed{\text{OVERESTIMATE}}$ UNDERESTIMATE NOT ENOUGH INFORMATION

4. [15 points] On the axes below, part of the graph of a **continuous** function $f(x)$ is given. Suppose $f(x)$ has the following properties:

- $f(x)$ is **piecewise linear** on $[-3, 5]$.
- $\int_{-5}^{-3} f'(x)\,dx = \frac{3}{2}$.
- $\int_{-3}^{0} f(x)\,dx = 3$.
- $\int_{0}^{3} f'(x)\,dx = 2$.
- The average value of $f(x)$ on $[3, 5]$ is 1.

Sketch the rest of a possible graph of $f(x)$ on $[-5, 5]$, labeling all x and y intercepts. Label the x and y coordinates of the points on the graph at $x = 3, 5$, and also label the y coordinate of the point at $x = -5$. Be sure all other important features of your graph are clear.

Solution: One possible graph is shown below.



5. [8 points] In each part of this problem, **circle all correct options**. There may be more than one correct answer for each part. If none of the options are correct, then circle NONE OF THESE. You must circle your answers entirely to receive credit on this problem, but no work is necessary.

a. [4 points] Which of the following are antiderivatives to the function $G(x) = e^{x^2}$?

$$\int_0^{x^2} e^t \, dt \qquad \boxed{\int_{100}^{x} e^{t^2} \, dt} \qquad \frac{e^{x^2}}{2x} \qquad \frac{e^{x^2}}{2x} - \frac{e^{x^2}}{2}$$

NONE OF THESE

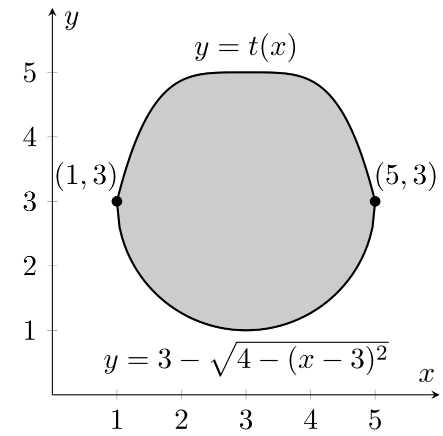
b. [4 points] Consider a solid whose base is the region bounded by the line $y = 2 - 2x$, the x-axis and the y-axis, and whose cross sections perpendicular to the x-axis are rectangles with their base being half of their height. Which of the following are equal to the volume of the solid?

$$\int_0^2 2\left(1 - \frac{y}{2}\right)^2 dy \qquad \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy$$

$$\boxed{\int_0^1 2(2-2y)^2 \, dy} \qquad \int_0^1 (2-2y)^2 \, dy$$

NONE OF THESE

6. [16 points] Ariana, a baker at the vegan bakery VCorp, is designing a new doughnut. The cross section of the doughnut is shown below, where the units of both x and y are cm.



The top of the cross section is given by the function $y = t(x)$ and the bottom is given by the semicircle $y = 3 - \sqrt{4-(x-3)^2}$. Ariana is experimenting with two ideas for the doughnut.

a. [6 points] Her first idea is to rotate the cross section around the y-axis. Write an integral that gives the volume of the resulting doughnut. Do not evaluate your integral. Your answer may involve the function t, but it should not involve t^{-1} (the inverse of t).

Solution: We use the shell method here. For each vertical slice of thickness Δx (cm), we have $\Delta V = 2\pi x(t(x) - (3 - \sqrt{4-(x-3)^2}))\Delta x$ (cm^3). The volume of the resulting doughnut (in cm^3) is given by integrating the above expression from $x = 1$ to $x = 5$.

Answer: $\displaystyle\int_1^5 2\pi x(t(x) - (3 - \sqrt{4-(x-3)^2}))\, dx$ $\textbf{cm}^3$.

b. [6 points] Her second idea is to rotate the cross section around the x-axis. Write an integral that gives the volume of the resulting doughnut. Do not evaluate your integral. Your answer may involve the function t, but it should not involve t^{-1} (the inverse of t).

Solution: We use the washer method here. For each vertical slice of thickness Δx (cm), we have $\Delta V = \pi((t(x))^2 - (3 - \sqrt{4-(x-3)^2})^2)\Delta x$ (cm^3). The volume of the resulting doughnut (in cm^3) is given by integrating the above expression from $x = 1$ to $x = 5$.

Answer: $\displaystyle\int_1^5 \pi((t(x))^2 - (3 - \sqrt{4-(x-3)^2})^2)\, dx$ $\textbf{cm}^3$.

c. [4 points] As is tradition at VCorp, they are planning to wrap a ribbon around the cross section of the doughnut. Write an expression involving one or more integrals that gives the total perimeter of the cross section. Do not evaluate the integrals in your expression.

Solution: We find the length of the top and bottom halves of the cross section separately, and add them. For the top curve, $y' = t'(x)$. And, for the bottom curve, $y' = \dfrac{x-3}{\sqrt{4-(x-3)^2}}$. Therefore, using the arclength formula $\displaystyle\int_{x=a}^{x=b} \sqrt{1+(y')^2}\,dx$, with $a = 1$ and $b = 5$ for each of these pieces and adding them, we get the total perimeter (in cm) of the cross section,

$$\int_1^5 \sqrt{1+(t'(x))^2}\,dx + \int_1^5 \sqrt{1+\left(\frac{x-3}{\sqrt{4-(x-3)^2}}\right)^2}\,dx.$$

The length of the bottom piece can also be directly calculated to be 2π, by noting that it is a semicircle of radius 2.

Answer: $\displaystyle\int_1^5 \sqrt{1+(t'(x))^2}\,dx + 2\pi$ **cm**

7. [10 points] Jamal is refurbishing a watch and he is placing small white sapphires around the watch face. The existing space on the watch face can accept regularly shaped sapphires of volume (in cubic mm):

$$V(t) = \int_{-\frac{t}{2}}^{1-t} \sqrt{1-x^2}\, dx$$

for any number t satisfying $0 \le t \le 1$.

a. [4 points] Compute $V'(t)$. Your expression should not involve any integrals.

Solution: By the chain rule and the second FTC, we have

$$V'(t) = \sqrt{1-(1-t)^2} \cdot (-1) - \sqrt{1-\left(-\frac{t}{2}\right)^2} \cdot \left(\frac{-1}{2}\right).$$

Answer: $V'(t) =$ $-\sqrt{1-(1-t)^2} + \frac{1}{2}\sqrt{1-\left(-\frac{t}{2}\right)^2}$.

b. [6 points] Jamal would like to know the volume of the smallest sapphire he can use on the watch face. Given that $V(t)$ has its only critical point at $t \approx \frac{2}{15}$, find the t-value(s) in the interval $0 \le t \le 1$ where the minimum of $V(t)$ occurs. Justify your answer using the fact that the graph of $y = \sqrt{1-x^2}$ is the top half of the unit circle centered at the origin.

Solution: First, we recall that the *global* minimum of $V(t)$ on $0 \le t \le 1$ can only occur at any critical point(s) in the interval or at the endpoint(s) of the interval.

As we are provided that $t \approx \frac{2}{15}$ is the only critical point, we will first attempt to classify it as a local min/max. For this, we invoke the first derivative test with the value of V' on either side of $t \approx \frac{2}{15}$, say at $t = 0$ and $t = 1$ for computational convenience. Using our result in part **a.**, we calculate $V'(0) = \frac{1}{2} > 0$, and $V'(1) = -1 + \frac{1}{2}\sqrt{\frac{3}{4}} < 0$. Therefore, by the first derivative test, $V(t)$ has a local maximum at $t \approx \frac{2}{15}$. Therefore, it cannot be a global minimum.

The only other candidates, for the global minimum, that remain are the endpoints of the interval, namely $t = 0$ and $t = 1$. We note that $V(0) = \int_0^1 \sqrt{1-x^2}\, dx$ is the area of quarter of the unit circle, whereas $V(1) = \int_{-\frac{1}{2}}^{0} \sqrt{1-x^2}\, dx$ describes an area that is less than that of the quarter unit circle. Therefore, $V(1) < V(0)$, and hence, $t = 1$ must be the global minimum of $V(t)$.

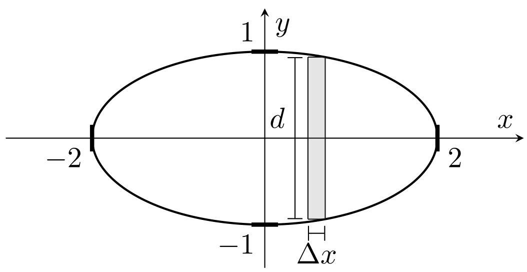
Alternatively, we can directly compare the values $V(0)$, $V(\frac{2}{15})$, and $V(1)$.

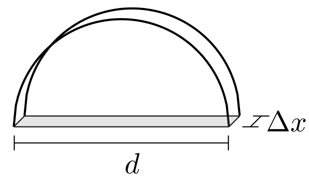
- $V(0) = \int_0^1 \sqrt{1-x^2}\, dx$ is the area of quarter of the unit circle.
- $V(1) = \int_{-\frac{1}{2}}^{0} \sqrt{1-x^2}\, dx$ describes an area that is less than that of the quarter unit circle.
- $V(\frac{2}{15}) = \int_{-\frac{2}{30}}^{\frac{13}{15}} \sqrt{1-x^2}\, dx$ includes the interval $[0, \frac{1}{2}]$ which, due to symmetry, implies that $V(\frac{2}{15})$ describes an area that is larger than $V(1)$.

As $V(1)$ is smaller than both $V(0)$ and $V(\frac{2}{15})$, by the Extreme Value Theorem, $t = 1$ must be the global minimum of $V(t)$.

Answer: The minimum of $V(t)$ occurs at $t =$ **1**.

8. [12 points] Astronomers have spotted a small near-Earth asteroid hurtling towards Earth. In order to assess its danger, they set about calculating its mass. Based on telescope images, the base of the asteroid is given by the region enclosed in the figure on the left, and its cross-sections perpendicular to the x-axis are semi-circles (as shown in the figure on the right). The base is the region bounded by $\frac{x^2}{4}+y^2=1$. A sample slice of the base of thickness Δx is shown in graph on the left, and all distances are given in meters.





a. [3 points] Write an expression for the diameter, d, in meters, of a cross-sectional slice of the asteroid x meters from the y-axis.

Answer: $d =$ $2\sqrt{1-\dfrac{x^2}{4}}$ **m**.

b. [4 points] Write an expression for the volume, V, in m^3, of a cross-sectional slice of the asteroid x meters from the y-axis with thickness Δx meters.

Answer: $V =$ $\dfrac{\pi}{2}\left(1-\dfrac{x^2}{4}\right)\Delta x$ $\mathbf{m}^3$.

c. [2 points] The density of the asteroid depends on x due to shearing (i.e. loss of material) in its direction of travel. The astronomers have computed the expression for the density of a cross-sectional slice of the asteroid to be $\delta(x)=\frac{4000}{7}(x+2)$ kg/m^3. What is the mass, $m(x)$, in kg, of a cross sectional slice of the asteroid with thickness Δx meters?

Answer: $m(x) =$ $\dfrac{\pi}{2}\left(1-\dfrac{x^2}{4}\right)\left(\dfrac{4000}{7}(x+2)\right)\Delta x$ **kg**.

d. [3 points] Write an integral that gives the total mass of the asteroid in kg. Do not evaluate your integral.

Answer: Total Mass = $\displaystyle\int_{-2}^{2}\frac{\pi}{2}\left(1-\frac{x^2}{4}\right)\left(\frac{4000}{7}(x+2)\right)dx$ **kg**.