Math 116 — Second Midterm — March 20, 2023

Write your 8-digit UMID number very clearly in the box to the right.

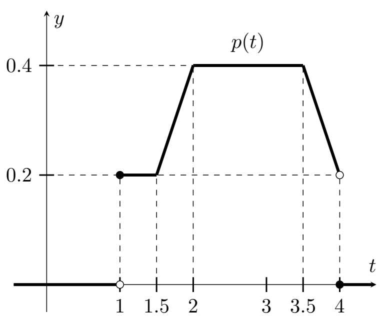
Your Initials Only: ______ Instructor Name: ________________________ Section #: ____

1. This exam has 11 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
5. You are allowed notes written on two sides of a $3'' \times 5''$ note card.
6. You are NOT allowed other resources, including, but not limited to, notes, calculators or other devices.
7. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
8. Include units in your answer where that is appropriate.
9. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
10. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	15	
2	14	
3	12	
4	15	
5	8	
6	8	

Problem	Points	Score
7	16	
8	12	
Total	100	

1. [15 points] There are 1000 contestants taking part in the reality show *Endure and Survive*. In the show's first contest, the contestants must complete a timed run of a 500-meter obstacle course. Those who finish within 2 minutes avoid elimination. At the conclusion of this game, the show's director creates a **probability density function** $p(t)$ (shown in the graph below) to model the distribution of the finishing times for the 1000 contestants, where t is given in minutes.



a. [4 points] What was the median finishing time for the 1000 contestants according to the director's model? Include units.

Solution: We find the value M, such that $\int_{-\infty}^{M} p(t)\,dt = 0.5$. First, note $p(t) = 0$, for $t < 1$.
Next, note that $\int_1^2 p(t)\,dt = 1 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.5 = 0.2 + 0.05 = 0.25$. In other words, the area accumulated under $p(t)$ until $t = 2$ is 0.25. Now, note that $\int_2^{3.5} p(t)\,dt = 0.4 \cdot 1.5 = 0.6$. Therefore, the area accumulated under $p(t)$ until $t = 3.5$ is $0.25 + 0.6 = 0.85$. This tells us that $2 < M < 3.5$. Given this, we need

$$\begin{aligned} 0.5 &= \int_1^M p(t)\,dt \\ &= \int_1^2 p(t)\,dt + \int_2^M p(t)\,dt \\ &= 0.25 + 0.4(M-2) \\ 0.25 &= 0.4(M-2) \\ M &= 2 + \frac{0.25}{0.4} = 2 + \frac{5}{8} = \frac{21}{8} = 2.625. \end{aligned}$$

Therefore, the median finishing time for the 1000 contestants, according to the director's model, was 2.625 minutes.

Answer: 2.625 minutes

b. [2 points] According to the director's model, how many of the 1000 contestants were eliminated in this game?

Solution: According to the director's model, the fraction of contestants eliminated is given by

$$\int_2^{\infty} p(t)\,dt = 1 - \int_1^2 p(t)\,dt = 1 - 0.25 = 0.75.$$

Therefore, $0.75 \cdot 1000 = 750$ contestants were eliminated in this game.

Answer: 750 contestants

1. (continued) For the next segment of the show, each remaining contestant goes through the obstacle course again. This time, their goal is to travel as much distance within the obstacle course as they can in one minute. The **probability density function**, $r(x)$, for the amount of distance x (measured in hundreds of meters) the remaining contestants travel within the obstacle course in one minute is given by

$$r(x) = \begin{cases} \frac{1}{30}(3x^2 - 3) & \text{for } 1 < x < 3, \\ a(x-5)^2 & \text{for } 3 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

c. [6 points] Find the value of a so that $r(x)$ is a probability density function.

Solution: For $r(x)$ to be a probability density function, it must satisfy $\int_{-\infty}^{\infty} r(x)\,dx = 1$. Since $r(x)$ is only nonzero on the interval $1 < x < 5$, we need it to satisfy,

$$\int_1^3 \frac{1}{30}(3x^2 - 3)\,dx + \int_3^5 a(x-5)^2\,dx = 1.$$

Solving, we get

$$\begin{aligned} \frac{1}{30}(x^3 - 3x)\Big|_1^3 + \frac{a(x-5)^3}{3}\Big|_3^5 &= 1 \\ \frac{1}{30}(27 - 9 - (1-3)) + a(0 - (-\frac{8}{3})) &= 1 \\ \frac{2}{3} + \frac{8}{3}a = 1 &\implies a = \frac{1}{8}. \end{aligned}$$

Answer: $a = \underline{\mathbf{\frac{1}{8}}}$

d. [3 points] Below, circle the **one** best interpretation of the equation

$$r(2) = 0.3.$$

- 30% of the remaining contestants travel exactly 200 meters within the obstacle course in one minute.
- Approximately 30% of the remaining contestants travel exactly 200 meters within the obstacle course in one minute.
- 60% of the remaining contestants travel at most 200 meters within the obstacle course in one minute.
- Approximately 1.5% of the remaining contestants travel between 200 meters and 205 meters within the obstacle course in one minute.
- Approximately 1.5% of the remaining contestants travel between 195 meters and 205 meters within the obstacle course in one minute.
- Approximately 2% of the remaining contestants travel exactly 300 meters within the obstacle course in one minute.

2. [14 points] Consider the following sequences, all defined for $n = 1, 2, 3, \ldots$

$$a_n = \int_0^n 10e^{-t}\,dt$$
$$b_n = (-1)^n \frac{100}{n^{0.75}}$$
$$c_n = 5(-3)^{n-3}$$

a. [3 points] Which sequences are monotone? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answer(s): $\boxed{a_n}$ b_n c_n NONE

b. [3 points] Which sequences are bounded? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answer(s): $\boxed{a_n}$ $\boxed{b_n}$ c_n NONE

c. [3 points] Which sequences are convergent? No justification is required for this part of the problem. Circle your final answer(s) below.

Circle your answer(s): $\boxed{a_n}$ $\boxed{b_n}$ c_n NONE

d. [5 points] Write a closed form expression for the series $\sum_{n=2}^{2023} c_n$. Your expression should be able to be evaluated using a simple calculator (i.e. no letters, no ellipses $(\ldots)$ and no sigma notation). Do not simplify the numbers in your expression.

Solution:

- The first term in the series is $c_2 = 5(-3)^{2-3} = -\frac{5}{3}$.
- The number of terms in the series is 2022.
- The common ratio between consecutive terms in the series is -3.

Using these three facts, and the formula for the sum of a finite geometric series, we obtain the answer below.

Answer: $\sum_{n=2}^{2023} c_n = -\frac{5}{3}\left(\frac{1-(-3)^{2022}}{1-(-3)}\right)$

3. [12 points] Consider the following sequences, all defined for $n = 1, 2, 3, \ldots$

$$a_n = \int_0^n 10e^{-t}\,dt$$
$$b_n = (-1)^n \frac{100}{n^{0.75}}$$

These are the same first two sequences from the previous problem.

a. [6 points] Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Fully justify your answer, including full mechanics of any tests you use.

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Solution: Consider

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int_0^n 10e^{-t}\,dt = \lim_{n\to\infty} -10e^{-t}\Big|_0^n = \lim_{n\to\infty} -10e^{-n} + 10 = 10 \neq 0.$$

Therefore, by the Divergence Test, $\sum_{n=1}^{\infty} a_n$ must diverge.

b. [6 points] Does the series $\sum_{n=1}^{\infty} b_n$ converge or diverge? Fully justify your answer, including full mechanics of any tests you use.

Circle one: $\boxed{\textbf{Converges}}$ **Diverges**

Solution: Consider $d_n = \frac{100}{n^{0.75}}$. Since $d_n > 0$ (i.e. d_n is positive), and $d_{n+1} = \frac{100}{(n+1)^{0.75}} < \frac{100}{n^{0.75}} = d_n$ (d_n is decreasing) for all n, and

$$\lim_{n\to\infty} d_n = \frac{100}{n^{0.75}} = 0,$$

by the Alternating Series Test, we must have that $\sum_{n=1}^{\infty} (-1)^n d_n = \sum_{n=1}^{\infty} b_n$ must converge.

4. [15 points] For each of the following, circle ALL that apply. There may be more than one correct answer for each part. You do not need to show any work for any part of this question.

a. [5 points] Suppose $P(t)$ is a **cumulative distribution function** (cdf) satisfying $P(0.5) = 0.4$. Which of the following MUST be true?

$\boxed{\lim_{t\to\infty} P(t) = 1.}$

0.4 is a median t-value for the distribution.

$\boxed{P(0.4) \leq 0.5.}$

The mean t-value of the distribution is 0.4.

NONE OF THESE

b. [5 points] A building is flooded, completely filled with water. The interior of the building is in the shape of a cube with side length 4 meters. To restore it, water is pumped out of the building to a temporary reservoir that lies **1 meter above the top of the building**. The density of water is 1000 kg per cubic meter. The acceleration due to gravity is g, where $g = 9.8\,\mathrm{m/s^2}$. Which of the following are equal to the work done, in Joules, in this pumping process?

$\int_0^4 16 \cdot h \cdot 1000 \cdot 9.8\,dh$

$\boxed{\int_0^4 16 \cdot (1+h) \cdot 1000 \cdot 9.8\,dh}$

$\boxed{\int_0^4 16 \cdot (5-h) \cdot 1000 \cdot 9.8\,dh}$

$\int_0^5 16 \cdot h \cdot 1000 \cdot 9.8\,dh$

NONE OF THESE

c. [5 points] The integral $\int_2^\infty \frac{x^{2/3}}{x+x^2}\,dx \ldots$

Diverges by the comparison test because $\frac{x^{2/3}}{x+x^2} \geq \frac{1}{x^{1/3}}$ for $x \geq 2$.

$\boxed{\text{Converges by the comparison test because } \frac{x^{2/3}}{x+x^2} \leq \frac{1}{x^{4/3}} \text{ for } x \geq 2.}$

Diverges because $\frac{x^{2/3}}{x+x^2} > 0$ for $x \geq 2$.

Converges because $\lim_{x\to\infty} \frac{x^{2/3}}{x+x^2} = 0$.

NONE OF THESE

5. [8 points] Use the **comparison test** for series to determine if the following series converges or diverges. **Circle your final answer choice**. Fully justify your answer including using proper notation and showing mechanics of the comparison test.

$$\sum_{n=3}^{\infty} \frac{\ln(n)}{n+\ln(n)}$$

Circle one: **Converges** $\boxed{\textbf{Diverges}}$

Solution: We first note that $\ln n < n$ for all n, and $\ln n > 1$ for all $n \geq 3$. Therefore, we have

$$\frac{\ln n}{n+\ln n} > \frac{1}{n+\ln n} > \frac{1}{n+n} = \frac{1}{2n}, \text{ for } n \geq 3.$$

Next, by the p-test ($p = 1$), we know that $\sum_{n=3}^{\infty} \frac{1}{2n}$ diverges. Therefore, by the (direct) comparison test (along with the inequality above, and the fact that the terms of the original series are positive), we have that $\sum_{n=3}^{\infty} \frac{\ln(n)}{n+\ln(n)}$ must also diverge.

6. [8 points] Determine whether the following series converges or diverges. If it converges, determine if it is absolute or conditional convergence. **Circle your final answer choice.** Fully justify your answer including using proper notation and showing mechanics of any tests you use.

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2-2}$$

Circle one: $\boxed{\textbf{Absolutely Converges}}$ **Conditionally Converges** **Diverges**

Solution: We first investigate absolute convergence, i.e. the convergence/divergence of the absolute value series $\displaystyle\sum_{n=3}^{\infty} \left|\frac{(-1)^n}{n^2-2}\right| = \sum_{n=3}^{\infty} \frac{1}{n^2-2}$.

Note that $n^2 - 2 \geq \frac{1}{2}n^2$, for $n \geq 3$. Therefore,

$$\frac{1}{n^2-2} \leq \frac{2}{n^2}, \text{ for } n \geq 3.$$

Now, by the p-test ($p = 2$), we know that $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges. Therefore, by the (direct) comparison test (along with the inequality above, and the fact that n^2-2 is positive), we have that $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-2}$ must also converge. So, by the Absolute Convergence Test, since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2-2}$ converges, we have that $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2-2}$ (absolutely) converges.

7. [16 points] A treasure hunter has spotted a large exotic rock at the bottom of a deep pit. The vertical distance from the top of the pit to the top of the rock is 15 meters. To retrieve the rock, the treasure hunter attaches a 15 meter rope to the top of the rock and lifts it out of the pit. The rope used has mass 2 kg per meter. Below, **do not simplify your final answers or evaluate any integrals**. As a reminder, the acceleration due to gravity is g, where $g = 9.8\,\mathrm{m/s}^2$.

a. [8 points] If the rock has mass 4 kg, write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the rock and the attached rope 10 meters up from the bottom of the pit.

Hint: Once rope has been raised to the top of the pit, the treasure hunter no longer needs to lift it.

Solution: After lifting the rock and the attached rope a distance of h meters, the length of the attached rope yet to be lifted is $15 - h$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$4 + 2 \cdot (15 - h) \text{ kg.}$$

So, the work done to lift the rock and the attached rope a short distance Δh meters at this moment is,

$$(4 + 2 \cdot (15 - h)) \cdot 9.8 \cdot \Delta h \text{ Joules.}$$

Therefore, the amount of work done by the treasure hunter to lift the rock and the attached rope 10 meters up from the bottom of the pit is given by,

$$\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh \text{ Joules.}$$

Answer: $\int_0^{10} (4 + 2 \cdot (15 - h)) \cdot 9.8 \, dh$

b. [8 points] After the rock has been lifted 10 meters off the bottom of the pit, the rock starts to crumble, losing 0.1 kg of mass per second. The treasure hunter resumes lifting the rock at a constant speed of 0.5 meters per second. Write an expression involving integrals for the amount of work, in Joules, the treasure hunter does in lifting the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit.

*The hint from part **a.** still applies.*

Solution: Since the rock loses 0.1 kg of mass per second, and the treasure hunter lifts at a constant speed of 0.5 meters per second, we have that the rock loses mass at a rate of

$$\frac{0.1\frac{\text{kg}}{\text{s}}}{0.5\frac{\text{m}}{\text{s}}} = 0.2\frac{\text{kg}}{\text{m}}.$$

Now, after lifting the rock and the attached rope a further distance of x meters, the length of the attached rope yet to be lifted is $5 - x$ meters. Therefore, the combined mass of the rock and the attached rope being lifted at this moment is

$$(4 - 0.2x) + 2 \cdot (5 - x) \text{ kg}.$$

So, the work done to lift the rock and the attached rope a short distance Δx meters at this moment is,

$$((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \cdot \Delta x \text{ Joules}.$$

Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) the remaining 5 meters to the top of the pit is given by,

$$\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx \text{ Joules}.$$

Alternatively, we can compute quantities after we have lifted the rock and the attached rope for a duration of t seconds. The mass of the crumbling rock is $4 - 0.1t$ kg. Now, the attached rope here starts of with a mass of $2 \cdot 5 = 10$ kg, and every second 0.5 meters of it is being retracted/lifted. So, the mass of the attached rope is given by $10 - 2(0.5 \cdot t) = 10 - t$ kg. The combined mass of the rock and the attached rope is then

$$(4 - 0.1t) + (10 - t) \text{ kg}.$$

Now, in the next short period of Δt seconds, the system is lifted $0.5\Delta t$ meters. So, the work done to lift the rock and the attached rope in this short period is

$$((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5\Delta t \text{ Joules}.$$

Now, at a speed of 0.5 meters per second, it takes 10 seconds to lift the system the remaining 5 meters to the top. Therefore, the amount of work done by the treasure hunter to lift the crumbling rock (and the attached rope) in this process is given by,

$$\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt \text{ Joules}.$$

Answer: $\int_0^5 ((4 - 0.2x) + 2 \cdot (5 - x)) \cdot 9.8 \, dx$ **or** $\int_0^{10} ((4 - 0.1t) + (10 - t)) \cdot 9.8 \cdot 0.5 \, dt$

8. [12 points] The parts of this problem are unrelated to each other.

a. [5 points] Compute the following limit. Fully justify your answer including using proper notation.

$$\lim_{x\to\infty} 2x \ln\left(1+\frac{5}{x}\right)$$

Solution: We note that $\lim\limits_{x\to\infty} 2x = \infty$, and $\lim\limits_{x\to\infty} \ln\left(1+\frac{5}{x}\right) = 0$. So, the limit is in the "$\infty \cdot 0$" form, allowing us to attempt to use L'Hôpital's rule. So, we first express the limit as a fraction, and proceed as shown below.

$$\begin{aligned}
\lim_{x\to\infty} 2x \ln\left(1+\frac{5}{x}\right) &= 2\lim_{x\to\infty} \frac{\ln\left(1+\frac{5}{x}\right)}{\frac{1}{x}} \\
&\overset{\text{L'H}}{=} 2\lim_{x\to\infty} \frac{\frac{1}{1+\frac{5}{x}} \cdot \frac{-5}{x^2}}{\frac{-1}{x^2}} \\
&= 2\lim_{x\to\infty} \frac{1}{1+\frac{5}{x}} \cdot 5 \\
&= 10.
\end{aligned}$$

Answer: $\lim\limits_{x\to\infty} 2x \ln\left(1+\frac{5}{x}\right) = \underline{\mathbf{10}}$

b. [7 points] Compute the value of the following improper integral if it converges. If it does not converge, use a **direct computation** of the integral to show its divergence. **Circle your final answer choice.** Show your full computation, and use proper notation.

$$\int_0^3 \frac{1}{(x-3)^2}\,dx$$

Circle one: **Converges to:** ____________________ $\boxed{\textbf{Diverges}}$

Solution: We note that $\frac{1}{(x-3)^2}$ has a vertical asymptote at $x = 3$. Therefore, we express the given (improper) integral as follows, and proceed with direct computation.

$$\begin{aligned}
\int_0^3 \frac{1}{(x-3)^2}\,dx &= \lim_{b\to 3^-} \int_0^b \frac{1}{(x-3)^2}\,dx \\
&= \lim_{b\to 3^-} -\frac{1}{x-3}\Big|_0^b \\
&= \lim_{b\to 3^-} -\frac{1}{b-3} - \frac{1}{3} \\
&= +\infty \text{ (or DNE)}.
\end{aligned}$$

Therefore, by direct computation, $\int_0^3 \frac{1}{(x-3)^2}\,dx$ diverges.