Math 116 — First Midterm — February 12, 2024

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Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ____________________

Instructor Name: ____ Section #: ______________

1. This exam has 10 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	16	
2	15	
3	18	
4	9	

Problem	Points	Score
5	14	
6	11	
7	11	
8	6	
Total	100	

1. [16 points] Let $f(x)$ be a function that is **even** and **twice differentiable.** Some values of $f(x)$ and $f'(x)$ are given in the table below:

x	0	1	2	3	4
$f(x)$	-3	2	-1	0	5
$f'(x)$	0	4	$\sqrt{2}$	1	e

Use the table above to compute the **exact value** of the following integrals. If there is not enough information to determine the exact value of an integral, write "NEI." You need to evaluate all integrals completely, and your answers should not involve the letter f, but you do not need to simplify your final answers. Show all your work.

a. [3 points] $\displaystyle\int_{-2}^{2} f'(x)\,dx$

Answer: ______________________

b. [4 points] $\displaystyle\int_{1}^{e^2} \frac{f'(\ln(t))}{t}\,dt$

Answer: ______________________

c. [4 points] $\displaystyle\int_{1}^{3} (2w+1)f'(w)\,dw$

Answer: ______________________

d. [5 points] $\displaystyle\int_{1}^{2} 2x^3 f''(x^2)\,dx$

Answer: ______________________

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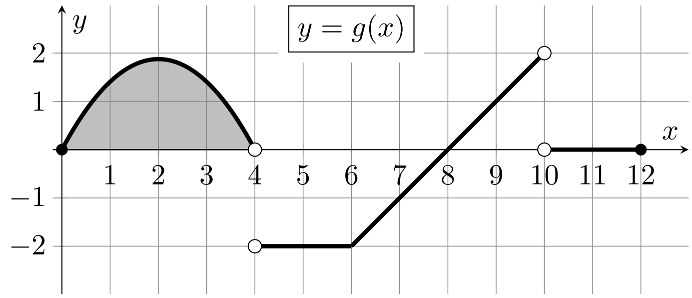
2. [15 points] A function $g(x)$ is graphed below and has the following properties:

- $g(x)$ is piecewise linear for $x > 4$.
- The shaded region has area 5.



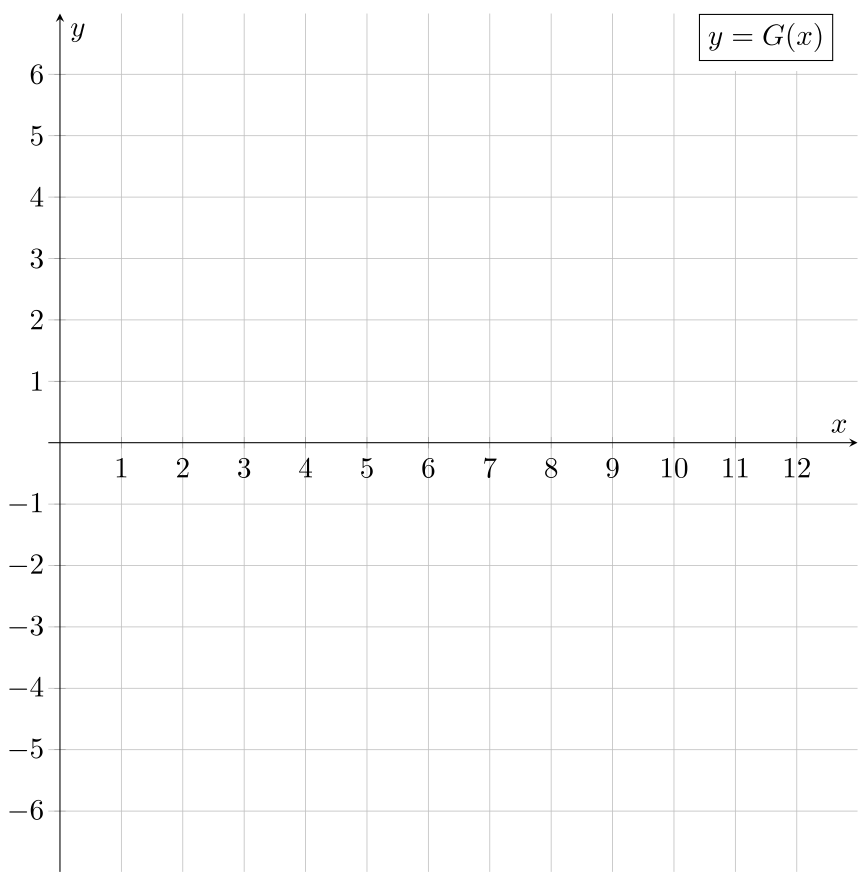
Let $G(x)$ be the continuous antiderivative of $g(x)$ satisfying $G(6) = -1$.

a. [5 points] Use the graph of $g(x)$ to complete the table below with the **exact** values of $G(x)$.

x	0	4	6	8	10	12
$G(x)$			-1			

b. [10 points] Sketch a graph of $G(x)$ on the interval $[0, 12]$ using the axes provided below. Be sure to pay attention to:

- where $G(x)$ is and is not differentiable;
- where $G(x)$ is increasing, decreasing, or constant;
- where $G(x)$ is concave up, concave down, or linear;
- the slope of $G(x)$ at $x = 2$;
- the values of $G(x)$ you found in the table in part **a.**



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3. [18 points] In the video game *The Legend of Helga*, the heroes Helga and Lank must save the world from the evil wizard Kanon. In the final battle, Helga and Kanon duel each other, while Lank creates magical spells to assist Helga.

a. [4 points] Using a magical spell costs Magic Points (MP), of which Lank has a limited amount. Let $L(t)$ be the amount of MP that Lank has remaining t minutes after the battle starts. The function $L(t)$ is differentiable, and its **derivative** $L'(t)$ is given by

$$L'(t) = (1 - t^3)^{1/3}.$$

Suppose that, after 5 minutes of battle, Lank has 20 MP remaining.
Write a formula involving an integral for $L(t)$. Your answer should not involve the letter L.

Answer: $L(t) =$ ______________________________

b. [5 points] While casting magical spells, Lank dodges Kanon's attacks by running along a path modeled by the curve

$$H(x) = \frac{1}{6}(4x - 1)^{3/2}$$

from $x = 1$ to $x = 16$, where distances are measured in meters.
Compute the **exact value** of the **arc length** of this curve. You need to evaluate all integrals completely, and your answer should not involve the letter H, but you do not need to simplify your final answer. Show all your work.

Answer: ______________________________ meters

This problem continues on the next page.

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3. (continued)

As Helga attacks Kanon throughout the battle, Kanon's Hit Points (HP) decrease. The amount of HP that Kanon has t minutes after the battle starts is given by the differentiable function

$$K(t) = \int_{t-4}^{\sqrt{t+16}} \left(12 + \cos(w^2)\right) dw.$$

c. [5 points] Compute the **exact value** of $K'(8)$. You do not need to simplify your final answer.

Answer: $K'(8) =$ ______

d. [4 points] Kanon is defeated upon reaching 0 HP, at which point the battle is over. Find the **one** value of t at which Kanon is defeated. Show all of your work.

Hint: If you find multiple such values of t, check which ones are actually solutions.

Answer: $t =$ ______

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4. [9 points] The city of Rainneapolis has a strange weather pattern. It is always sunny, except for February 2nd, when it rains substantially all day. This year, Amin prepared for the stormy day by building a machine which continuously removes rainwater from his backyard.

- Let $T(h)$ be the **total amount** of rainwater in Amin's backyard, in cubic feet, h hours after 12:00am on February 2nd.
- Let $A(h)$ be the **rate** at which the rain **adds** water to Amin's backyard, in cubic feet per hour, h hours after 12:00am on February 2nd.
- Let $M(h)$ be the **rate** at which Amin's **machine removes** rainwater from his backyard, in cubic feet per hour, h hours after 12:00am on February 2nd.

The functions $T(h)$, $A(h)$, and $M(h)$ are all differentiable. Assume that there is **no rainwater** in Amin's backyard **before** it starts raining at 12:00am on February 2nd.

a. [3 points] Which of the following gives a correct interpretation of $\int_4^{10} M(h)\,dh = 8000$? Circle **all** correct answers.

(i) The total amount of rainwater in Amin's backyard decreases by 8000 cubic feet from 4:00am to 10:00am on February 2nd.

(ii) Between 4:00am and 10:00am on February 2nd, Amin's machine removes a total of 8000 cubic feet of rainwater from his backyard.

(iii) The rate at which Amin's machine removes rainwater from his backyard between 4:00am and 10:00am on February 2nd is 8000 cubic feet per hour.

(iv) At 10:00am on February 2nd, Amin's machine removes rainwater from his backyard at a rate of 8000 cubic feet per hour faster than at 4:00am.

(v) NONE OF THESE

b. [3 points] Which of the following expressions gives the **total amount** of rainwater, in cubic feet, in Amin's backyard at 7:00am? Circle **all** correct answers.

(i) $\int_0^7 T'(h)\,dh$ (ii) $\int_0^7 T(h)\,dh$ (iii) $\int_0^7 \big(A(h) + M(h)\big)\,dh$

(iv) $\int_0^7 A(h)\,dh - \int_0^7 M(h)\,dh$ (v) NONE OF THESE

c. [3 points] Which of the following expressions gives the **average amount** of rainwater, in cubic feet, in Amin's backyard between 6:00am and 9:00am? Circle **all** correct answers.

(i) $\frac{1}{9-6}\int_6^9 T'(h)\,dh$ (ii) $\frac{1}{9-6}\int_6^9 T(h)\,dh$ (iii) $\frac{T(9) - T(6)}{9-6}$

(iv) $\frac{1}{3}\int_0^9 T(h)\,dh + \frac{1}{3}\int_6^0 T(h)\,dh$ (v) NONE OF THESE

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5. [14 points]

a. [6 points] Split the following expression into partial fractions with two or more terms. **Do not integrate these terms.** Please clearly show all of your work.

$$\frac{5x-4}{(x-2)^2(x+1)}$$

Answer: ____

b. [8 points] Use the partial fraction decomposition

$$\frac{4x-2}{(3-x)(x^2+1)} = \frac{1}{3-x} + \frac{x-1}{x^2+1}$$

to evaluate the following **indefinite** integral. Please clearly show all of your work.

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx.$$

Answer: ____

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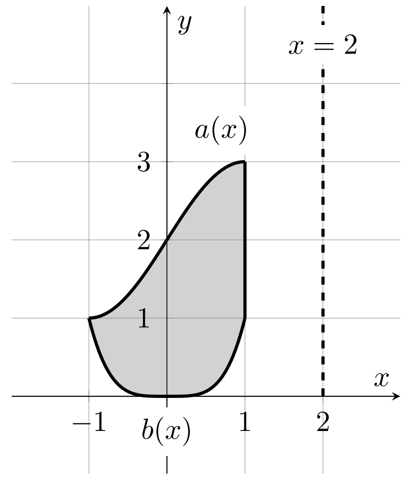
6. [11 points] Louise, a world-famous abstract artist and cheese enthusiast, is experimenting with new designs for cheese sculptures. She has two ideas for a cheese sculpture and would like to know the volume of each one so that she knows how much cheese to buy.

a. [6 points] Louise's first idea involves the shaded region to the right, which is bounded by the line $x = 1$ and the curves

$$a(x) = 2 + \sin\left(\frac{\pi}{2}x\right) \qquad \text{and} \qquad b(x) = x^4$$

on the interval $[-1, 1]$.

Write an integral that represents the volume of the solid formed by rotating this region around the line $x = 2$. **Do not evaluate your integral.** Your answer should not involve the letters a or b.



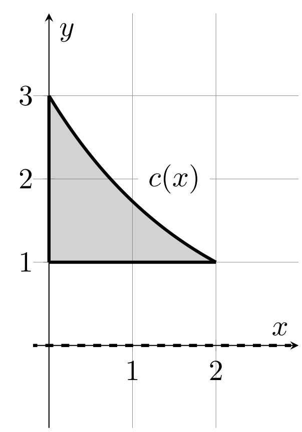
Answer: ______________________________

b. [5 points] Louise's second idea involves the shaded region to the right, bounded by the curve

$$c(x) = \left(\sqrt{3}\right)^{2-x},$$

the y-axis, and the line $y = 1$ on the interval $[0, 2]$.

Write an integral that represents the volume of the solid formed by rotating this region around the x-axis. **Do not evaluate your integral.** Your answer should not involve the letter c.



Answer: ______________________________

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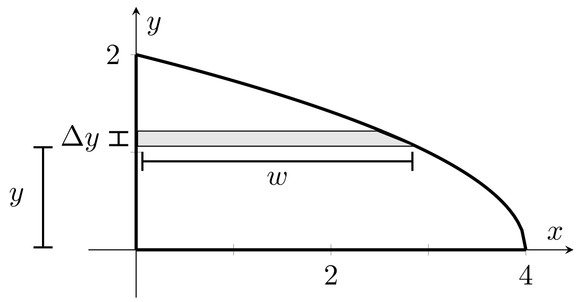
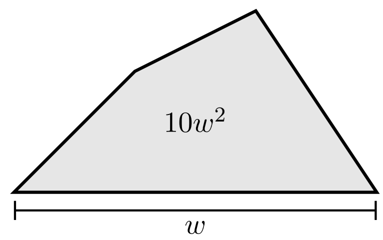
7. [11 points] In an accidental discovery, scientists created the Ultra Bouncy Toy (UBT), which bounces unpredictably due to its unusual shape and irregular density.

The base of the UBT is the region bounded by $y = \sqrt{4-x}$, the x-axis, and the y-axis, shown below to the left. All distances are measured in centimeters (cm). A sample slice of the base of width w and thickness Δy is shown in the graph below to the left. Cross-sections of the UBT perpendicular to the ***y*-axis** have the shape shown below to the right. The area of such a cross-section is $10w^2$.





a. [3 points] Write a formula in terms of y for the width w of a slice that is y centimeters above the x-axis. **Include units.**

Answer: $w =$ ____________________ **Units:** ________

b. [3 points] Write an expression that approximates the **volume of a slice** of the UBT that is y centimeters above the x-axis and has thickness Δy centimeters. Your answer should not involve the letter w. **Include units.**

Answer: ____________________ **Units:** ________

The density of the UBT is given by the function $\delta(y)$, measured in grams per cubic centimeter (g/cm^3), where y is the distance from the x-axis in centimeters.

c. [2 points] Write an expression that approximates the **mass of a slice** of the UBT that is y centimeters above the x-axis and has thickness Δy centimeters. Your answer may include δ, but it should not involve the letter w. **Include units.**

Answer: ____________________ **Units:** ________

d. [3 points] Write an expression involving an integral that represents the **total mass** of the UBT. Your answer may include δ. **Include units.**

Answer: ____________________ **Units:** ________

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8. [6 points] Each part below describes a twice differentiable function and one or more approximations of its integral. For each of the following statements, determine if the statement is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true, and circle the appropriate answer.
No justification is required.

a. [1 point] If $A'(x) > 0$ for all x, then $\text{LEFT}(4) \le \int_{-1}^{1} A(x)\,dx$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

b. [1 point] If $B'(x) > 0$ for all x, then $\text{TRAP}(4) \le \int_{-1}^{1} B(x)\,dx$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

c. [1 point] If $C''(x) > 0$ for all x, then $\text{TRAP}(4) \le \int_{-1}^{1} C(x)\,dx$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

d. [1 point] If $D(x)$ is odd and MID(4) approximates $\int_{-1}^{1} D(x)\,dx$, then $\text{MID}(4) = 0$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

e. [1 point] If $E'(x) > 0$ and $E''(x) < 0$ for all x, then $\int_{-1}^{1} E(x)\,dx \le \text{MID}(2) \le \text{RIGHT}(2)$.

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

f. [1 point] If $F(x)$ is not constant, then RIGHT(3) approximates the integral $\int_{-1}^{1} F(x)\,dx$ more accurately than RIGHT(2).

Circle one: **ALWAYS** **SOMETIMES** **NEVER**

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