Math 116 — Second Midterm — March 25, 2024

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ________________

Instructor Name: ________________________________ Section #: __________

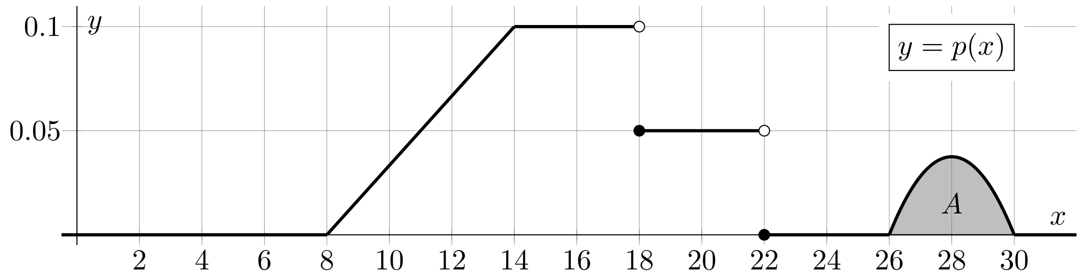
1. This exam has 10 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ notecard. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	12	
3	13	
4	11	
5	14	

Problem	Points	Score
6	10	
7	7	
8	6	
9	12	
10	6	
Total	100	

1. [9 points] Every Saturday during the summer, Dominic rides his bicycle in a national park. The distance he travels on his bicycle each Saturday varies.
Let $p(x)$ be the **probability density function** (pdf) for x, the distance (in miles) that Dominic bicycles on a Saturday. The graph of $p(x)$, shown below, has the following properties:

- $p(x)$ is piecewise linear for $x \leq 26$.
- $p(x)$ is nonzero only for $8 < x < 22$ and $26 < x < 30$.
- The area of the shaded region is A.



For each part of this problem, your answer should not involve the letter A. You do not need to show your work in this problem, but partial credit may be awarded for work shown clearly.

a. [1 point] Find the **minimum** distance that Dominic bicycles on a Saturday.

Answer: ____________________ miles

b. [2 points] Find the **median** distance that Dominic bicycles on a Saturday.

Answer: ____________________ miles

c. [2 points] Use the fact that $p(x)$ is a probability density function to find the value of A.

Answer: $A =$ ____________________

d. [2 points] Calculate the probability that Dominic bicycles farther than 18 miles on a Saturday.

Answer: ____________________

e. [2 points] Complete the sentence below to write a practical interpretation of the equation $p(28) = 0.0375$:

The probability that Dominic bicycles between 27 and 29 miles on a Saturday is...

2. [12 points] Joe and Paula are at the same national park, hiking through the forest. They arrive at the bottom of a cliff and challenge each other to bring their hiking gear to the top of the cliff, which is 25 meters above the bottom. Each of them has a different idea of how to accomplish this.
You may assume that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.

a. [6 points] Joe plans to climb to the top of the cliff while carrying his water bottle. Before Joe starts climbing, the combined mass of Joe and his water bottle is 64 kilograms. However, a rock punctures the bottle as soon as Joe starts climbing, so water leaks out at a constant rate of 0.03 kilograms per second. Joe climbs the cliff at a constant rate of 0.25 meters per second.

(i) Let $M(h)$ be the combined mass of Joe and his water bottle, in kilograms (kg), when Joe is h meters above the ground. Write an expression for $M(h)$.

Answer: $M(h) =$ ________________________

(ii) Write an integral representing the total amount of work, in Joules (J), that it takes for Joe to move himself and the water bottle to the top of the cliff. Your answer should not involve the letter M. **Do not evaluate your integral.**

Answer: ________________________

b. [6 points] Paula ties a rope to her 3-kilogram backpack, walks to the top of the cliff, and then uses the rope to pull her backpack to the top. The rope has a mass of 0.1 kilograms per meter. Write an integral representing the total amount of work, in Joules (J), that it takes for Paula to pull her backpack and the attached rope to the top of the cliff.
Do not evaluate your integral.

Answer: ________________________

3. [13 points] Consider the following sequences, each defined for $n \geq 1$:

$$a_n = \frac{\cos(\pi n)}{n} \qquad b_n = -\left(\frac{100}{99}\right)^n \qquad c_n = \sum_{k=0}^{n} \frac{1}{3^k}$$

a. [9 points] For each of the sequences above, determine whether the sequence is bounded, whether it is monotone, and whether it is convergent. No justification is required.

(i) The sequence a_n is...

Circle one: **Bounded** **Unbounded**

Circle one: **Monotone** **Not Monotone**

Circle one: **Convergent** **Divergent**

(ii) The sequence b_n is...

Circle one: **Bounded** **Unbounded**

Circle one: **Monotone** **Not Monotone**

Circle one: **Convergent** **Divergent**

(iii) The sequence c_n is...

Circle one: **Bounded** **Unbounded**

Circle one: **Monotone** **Not Monotone**

Circle one: **Convergent** **Divergent**

b. [4 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=0}^{\infty} c_n$$

Circle one: **Convergent** **Divergent**

4. [11 points] Zach is playing the retro video game *Plaque-Man* all day to get a new personal high score. Zach starts playing the game with 0 points. Over the course of each hour, Zach scores an additional 2500 points. At the **beginning** of every hour, Zach trades 20% of his points to buy extra time. For $n \geq 1$, let H_n be Zach's score at the **end** of the nth hour of playing the game.
For example, $H_1 = 2500$.

a. [4 points] Write expressions for H_2 and H_3. Your answers should not involve the letter H. You do not need to simplify your expressions.

$H_2 =$ ____________________

$H_3 =$ ____________________

b. [4 points] Write a **closed-form** expression for H_n. *Closed-form* means your answer should not include ellipses $(\ldots)$ or sigma notation (Σ), and should not be recursive. You do not need to simplify your closed-form expression.

Answer: $H_n =$ ____________________

c. [3 points] Find Zach's eventual score if he keeps playing *Plaque-Man* indefinitely. You do not need to simplify your numerical answer.

Answer: ____________________

5. [14 points]

a. [7 points] Determine whether the following improper integral is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. You do not need to compute the value of the integral if it is convergent. Circle your final answer choice.

$$\int_1^{\infty} \frac{4+\sin(x)}{x^3+2}\,dx$$

Circle one: **Convergent** **Divergent**

b. [7 points] Let $0 < p < 1$ be a real number, and consider the improper integral

$$\int_1^3 \frac{1}{t(\ln(t))^p}\,dt.$$

The integral above converges; to show this, **compute** its value. Your answer may involve p. Be sure to show your full computation, and be sure to use **proper notation.**
Remember: $0 < p < 1$.

Answer: $\displaystyle\int_1^3 \frac{1}{t(\ln(t))^p}\,dt =$ ______________________

6. [10 points] Liban is writing songs using a new style of music which he calls "new-age jazz." The longer that he spends writing a particular song, the better it turns out.

Let $Q(t)$ be the **cumulative distribution function** (cdf) for t, the number of days that it takes for Liban to write a particular song. The formula for $Q(t)$ is shown to the right, where $c > 0$ is a constant.

$$Q(t) = \begin{cases} 0 & t < 0, \\ \frac{c}{4}t^2 & 0 \le t \le 2, \\ 2c - c\,e^{2-t} & t > 2. \end{cases}$$

You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [3 points] Write a piecewise-defined formula for $q(t)$, the **probability density function** (pdf) corresponding to $Q(t)$. Your answer may involve c, but it should not involve the letter Q.

$$q(t) = \begin{cases} \rule{4cm}{0.4pt} & t < 0, \\ \rule{4cm}{0.4pt} & 0 \le t \le 2, \\ \rule{4cm}{0.4pt} & t > 2. \end{cases}$$

b. [3 points] Write an expression involving one or more integrals that represents the **mean** number of days that it takes for Liban to write a particular song. Your answer may involve c, but it should not involve the letters Q or q. **Do not evaluate your integral(s).**

Answer: ______________________________

c. [2 points] Use the fact that $Q(t)$ is a cumulative distribution function to find the value of c.

Answer: $c =$ ______________________________

d. [2 points] Circle the **one** correct answer below that completes the following sentence:

"The quantity $Q(5)$ represents...

(i) *...the probability that it takes exactly 5 days for Liban to write a song."*

(ii) *...the probability that it takes more than 5 days for Liban to write a song."*

(iii) *...the probability that it takes 5 days or less for Liban to write a song."*

(iv) *...the approximate probability that it takes between 4.5 and 5.5 days for Liban to write a song."*

(v) NONE OF THESE

7. [7 points] Determine whether the following series is convergent or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use.
Circle your final answer choice.

$$\sum_{n=2}^{\infty} \frac{4^n \cdot n^2}{n!}$$

Circle one: **Convergent** **Divergent**

8. [6 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

$$\lim_{x \to 0} \frac{e^{2x} - (x+1)^2}{\cos(x) - 1}$$

Answer: ____________________

9. [12 points] Determine whether the following series is absolutely convergent, conditionally convergent, or divergent. **Fully justify** your answer including using **proper notation** and **showing mechanics** of any tests you use. Circle your final answer choice.

$$\sum_{n=1}^{\infty} \frac{n \cdot (-1)^n}{n^2 + n^{1/2}}$$

Circle one: **Absolutely Convergent** **Conditionally Convergent** **Divergent**

10. [6 points] Let $f(x)$, $g(x)$, and $h(x)$ be continuous functions defined for $x > 0$ which also satisfy the following:

- $\dfrac{1}{x^2} \le f(x) \le \dfrac{1}{x^3}$ for all $0 < x < 1$, and $0 \le f(x) \le \dfrac{1}{e^x}$ for all $x > 4$.
- $\displaystyle\sum_{n=1}^{\infty} g(n)$ is absolutely convergent.
- $h(x)$ is differentiable, increasing, and concave down, and $\displaystyle\lim_{x\to\infty} h(x) = 1$.

Determine whether the following integrals and series are **CONVERGENT** or **DIVERGENT**, and circle the appropriate answer. If there is not enough information to decide, circle **NEI**.
No justification is required.

a. [1 point] $\displaystyle\int_0^{\infty} f(x)\,dx$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**

b. [1 point] $\displaystyle\int_2^{\infty} f(x)\,dx$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**

c. [1 point] $\displaystyle\int_1^{\infty} g(x)\,dx$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**

d. [1 point] $\displaystyle\sum_{n=1}^{\infty} \frac{1}{g(n)}$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**

e. [1 point] $\displaystyle\sum_{n=1}^{\infty} h'(n)$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**

f. [1 point] $\displaystyle\sum_{n=1}^{\infty} g(n)h(n)$

Circle one: **CONVERGENT** **DIVERGENT** **NEI**