Math 116 — Final Exam — April 25, 2024

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ______________________

Instructor Name: ____ Section #: ______________

1. This exam has 12 pages including this cover.
2. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. The last page of this exam is a formula sheet which you may remove. Please submit that page along with your exam. Work written on that page will not be graded.
4. Do not separate the other pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
7. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
8. You are allowed notes written on two sides of a $3'' \times 5''$ notecard. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
9. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
12. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	12	
3	9	
4	10	
5	12	

Problem	Points	Score
6	10	
7	8	
8	8	
9	9	
10	10	
Total	100	

1. [12 points] Let $g(x)$ be a **differentiable** function, and let $G(x)$ be a **continuous antiderivative** of $g(x)$. Some values of $g(x)$ and $G(x)$ are given in the table below:

x	-2	-1	0	1	2
$g(x)$	0	$\sqrt{3}$	4	5	-1
$G(x)$	π	$1/2$	-2	0	1

Use the table above to answer the following questions. Write your answers in **exact form**. If there is not enough information to complete a problem, write "NEI." Your answers should not involve the letters g or G, but you do not need to simplify your final answers. Show all your work.

a. [3 points] Compute the **average value** of $g'\big(g(x)\big) \cdot g'(x)$ on the interval $[-2, 2]$.

Answer: ____________________

b. [3 points] Compute $F'(1)$, where $F(x) = \int_{x^3-2}^{4} G(t)\,dt$.

Answer: ____________________

c. [3 points] Approximate $\int_{-2}^{2} G(x)\,dx$ using TRAP(2).

Answer: ____________________

d. [3 points] Compute $\lim_{x\to\infty} x\,G(1 + \frac{1}{x})$.

Answer: ____________________

2. [12 points] In the video game *Super Maria 64*, sisters Maria and Luisa travel through the Lilypad Kingdom to collect magical Rainbow Crystals. In the Sandland Desert, represented by the xy-plane, the sisters run around collecting all the Rainbow Crystals they see.
All distances in this problem are measured in kilometers. For $t \geq 0$, the sisters' positions t hours after they start running are given by the following parametric equations:

$$\text{Maria: } \begin{cases} x(t) = t^2 + t - 6 \\ y(t) = 2\sin(\pi t) \end{cases} \qquad \text{Luisa: } \begin{cases} x(t) = 2t^2 - 4t \\ y(t) = \cos(\frac{\pi}{2}t) \end{cases}$$

a. [2 points] Find **Maria's position** 1 hour after the sisters start running.

Answer: $x =$ ____________ $y =$ ____________

b. [3 points] Find **Maria's speed,** in kilometers per hour, 1 hour after the sisters start running.

Answer: ________________________

c. [3 points] Find **all** times $t \geq 0$ at which **Luisa** travels **directly north** (that is, not in any northwest or northeast direction). If there is no such time, write "NONE." Show your work to justify your answer.

Answer: $t =$ ____________________

d. [4 points] Find **all** times $t \geq 0$ at which Maria and Luisa are at the same position. If there is no such time, write "NONE." Show your work to justify your answer.

Answer: $t =$ ____________________

3. [9 points] The Taylor series centered at $x = 1$ for a function $T(x)$ is given by:

$$T(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(-5)^n \cdot (2n)!} (x-1)^{4n+3}.$$

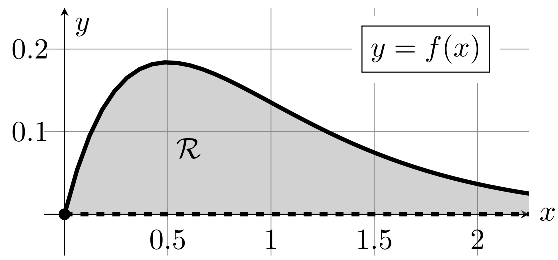
a. [6 points] Find the **radius of convergence** of the Taylor series above. Show your work. Do not attempt to find the interval of convergence.

Answer: ______________________

b. [3 points] Compute $T^{(123)}(1)$. Show your work. You do not need to simplify your answer.

Answer: $T^{(123)}(1) =$ ______________________

4. [10 points] Louise, the world-famous abstract artist and cheese enthusiast, has a dream about an infinitely-long cheese sculpture. The sculpture involves $\mathcal{R}$, which is the region above the x-axis and below the curve $f(x) = xe^{-2x}$ on the interval $[0, \infty)$. A portion of $\mathcal{R}$ is the shaded region below.



a. [4 points] Write an improper integral that represents the **volume** of the infinitely-long solid of revolution formed by rotating the region $\mathcal{R}$ around the x-axis.
Your answer should not involve the letter f. **Do not evaluate your integral.**

Answer: ______________________________

b. [6 points] The **area** of the region $\mathcal{R}$ (not the volume of the rotated solid) is given by the improper integral

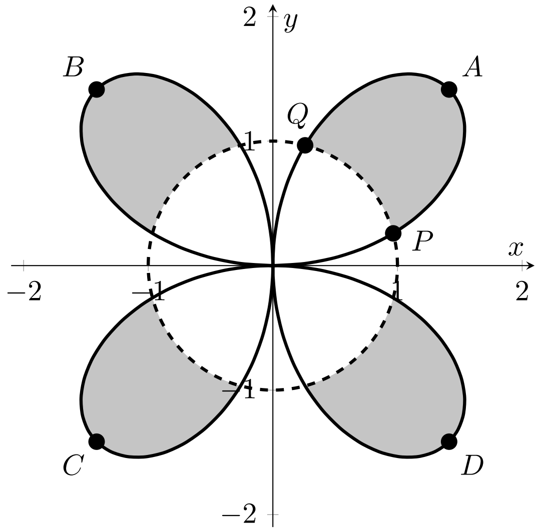
$$\int_0^\infty xe^{-2x}\, dx.$$

Determine whether this improper integral is convergent or divergent.
You may use either a direct computation or the comparison test to reach your conclusion.
Fully justify your answer including using **proper notation.** Circle your final answer choice.

Circle one: **Convergent** **Divergent**

5. [12 points] A large four-leaf clover, pictured below, resides in a forest.

- The leaves of the clover are modeled by the polar curve
$$r = 2\sin(2\theta)$$
for $0 \leq \theta \leq 2\pi$. This is the **solid** curve in the diagram to the right.
- The polar curve
$$r = 1$$
for $0 \leq \theta \leq 2\pi$ is the **dashed** curve in the diagram to the right.
The leaves of the clover are light green inside of this curve, and dark green outside of it.
- All distances are measured in inches.



a. [2 points] Which of the following points labelled in the diagram above is in the portion of the polar curve $r = 2\sin(2\theta)$ traced out for $\frac{\pi}{2} \leq \theta \leq \pi$? Circle the **one** correct answer.
No justification is required.

Circle one: *A* *B* *C* *D* NONE OF THESE

b. [5 points] The points P and Q, labelled above, are two intersection points of the solid and dashed curves. Write P and Q in **polar coordinates** $(\boldsymbol{r}, \boldsymbol{\theta})$, where $r \geq 0$ and $0 \leq \theta \leq 2\pi$.
Please show all of your work.

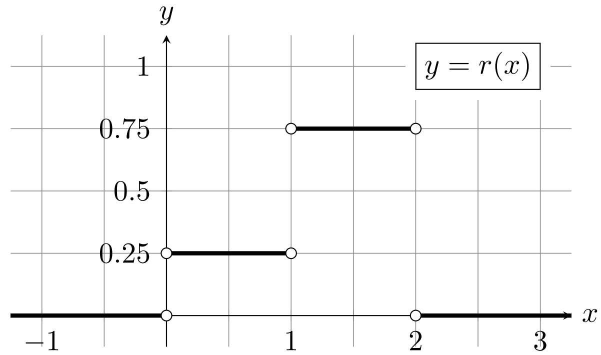
Answer: P: $(r, \theta) =$ ______________ Q: $(r, \theta) =$ ______________

c. [5 points] Write an expression involving **at most two integrals** that gives the **area**, in square inches, of the dark green part of the four-leaf clover. (This is the shaded region in the diagram above.) **Do not evaluate your integral(s).**

Answer: ______________

6. [10 points]
A survey has recently been conducted on the University of Michigan campus which asked a large number of students to choose a random real number in the interval $[0, 2]$.

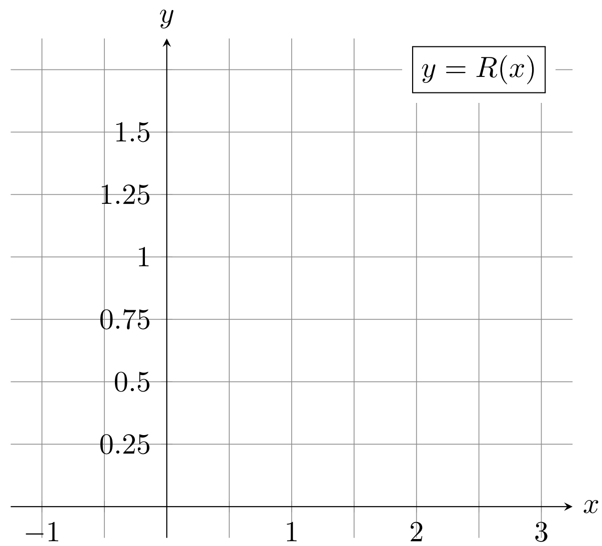
The numbers chosen by students are described by the **probability density function** (pdf) $r(x)$. A graph of $r(x)$ is shown to the right.



You do not need to show your work in this problem, but partial credit may be given for work shown.

a. [5 points]
Let $R(x)$ be the **cumulative distribution function** (cdf) corresponding to $r(x)$. The function $R(x)$ is defined for all real numbers x.
On the axes provided to the right, sketch a graph of $R(x)$ **on the interval** $[-1, 3]$.
Be sure to pay attention to:

- where $R(x)$ is and is not differentiable;
- where $R(x)$ is increasing, decreasing, or constant;
- where $R(x)$ is concave up, concave down, or linear;
- the values of $R(x)$ at $x = -1, 0, 1, 2$, and 3.



b. [2 points] Compute the fraction of students that chose a number in the interval $[1, 2]$.

Answer: ______________________

c. [3 points] Compute the **median** of the numbers chosen among all students.

Answer: ______________________

7. [8 points] Consider the power series below, centered at $x = 2$:

$$\sum_{n=1}^{\infty} \frac{n+2}{4^n \cdot n^3} (x-2)^n.$$

Its radius of convergence is 4; you do not need to show this.
Find the **interval of convergence** of this power series. Fully justify your answer including using proper notation and showing mechanics of any tests you use.

Answer: ______________________

8. [8 points]

a. [4 points] Write down the first 3 nonzero terms of the Taylor series for the function

$$S(x) = \begin{cases} \dfrac{e^{x^2} - 1}{3x^2} & x \neq 0, \\ \dfrac{1}{3} & x = 0, \end{cases}$$

centered at $x = 0$. You do not need to simplify any numbers in your answer.

Answer: ______________________________

b. [4 points] Compute the following limit. **Fully justify** your answer including using **proper notation.**

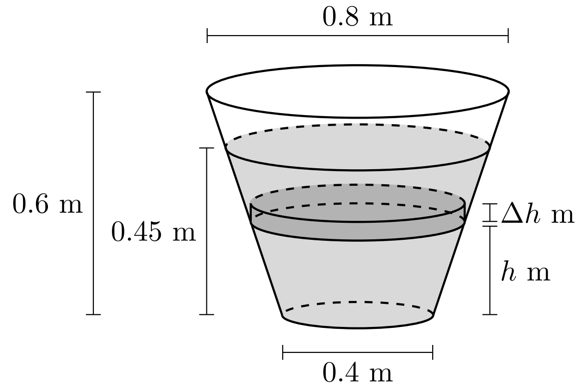
$$\lim_{x \to 0} \frac{\displaystyle\int_0^x \left(e^{t^2} - 1\right) dt}{x^3}$$

Hint: Your answer from the previous part may be helpful at some point.

Answer: __________

9. [9 points] Over the course of the summer, Markie makes many pitchers of their favorite lemonade. The special ingredient is a mixture of different types of sugar, which Markie scoops out of the extremely large bowl pictured below.

- The top of the bowl is circular and has a **diameter** of 0.8 meters.
- The bottom of the bowl is circular and has a **diameter** of 0.4 meters.
- The height of the bowl is 0.6 meters.
- Initially, the bowl is only filled up to 0.45 meters from the bottom of the bowl.
- The sugar mixture has an uneven density of $100(9-4h)$ kg/m^3, where h is the distance above the bottom of the bowl, in meters.



As suggested by the picture above, the diameter of a circular cross-section is a linear function of h. In this problem, you may assume that the acceleration due to gravity is $g = 9.8$ m/s^2.

a. [5 points] Write an expression that approximates the **mass,** in kilograms, of **a slice** of sugar that is h meters above the bottom of the bowl and has small thickness Δh meters (as shaded in the picture above). Your answer should not involve any integrals.

Answer: ______________________________

b. [4 points]

(i) Write an expression that approximates the **amount of work,** in Joules, needed to lift **a slice** of sugar that is h meters above the bottom of the bowl and has small thickness Δh meters (as shaded in the picture above) to the **top of the bowl.**
Your answer should not involve any integrals.

Answer: ______________________________

(ii) Write an expression involving an integral representing the **total amount of work,** in Joules, needed for Markie to lift **all** the sugar to the **top of the bowl.**

Answer: ______________________________

10. [10 points] For each of the following parts, circle **all** correct answers. No justification is required.

a. [2 points] A power series $\sum_{n=0}^{\infty} A_n(x+1)^n$ **converges** at $x=-3$ and **diverges** at $x=2$.

At which of the following x-values, if any, **must** this power series **converge?**

$x=-4$ $x=-2$ $x=0$ $x=1$ $x=3$ NONE

b. [2 points] Another power series $\sum_{n=0}^{\infty} B_n(x+1)^n$ **converges** for all $x<-3$.

At which of the following x-values, if any, **must** this power series **converge?**

$x=-4$ $x=-2$ $x=0$ $x=1$ $x=3$ NONE

c. [2 points] Which, if any, of the following infinite series **converge** to $\frac{1}{2}$?

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ $\sum_{n=0}^{\infty} \frac{3}{8}\left(\frac{1}{4}\right)^n$ $\sum_{n=1}^{\infty} \frac{n^2+2}{2n^2}$ $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}$ NONE

d. [2 points] Consider the curve traced out by the parametric equations:

$$\big(x(t),\ y(t)\big) = \big(t^2,\ \sin(\pi t)\big) \qquad \text{for } t \geq 0.$$

Which, if any, of the following is the **slope** of the tangent line to this curve at $t=1$?

$\frac{2}{\pi}$ $-\frac{2}{\pi}$ $\frac{\pi}{2}$ $-\frac{\pi}{2}$ 2π -2π NONE

e. [2 points] Which, if any, of the following points given in **polar coordinates** $\boldsymbol{(r,\theta)}$ represent the same point as $(x,y)=(-1,0)$ in the xy-plane?

$(r,\theta)=(1,3\pi)$ $(r,\theta)=(1,\pi)$ $(r,\theta)=(-1,\pi)$ $(r,\theta)=(-1,0)$ NONE

“Known” Taylor Series (all around $x = 0$):

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots \qquad \text{for all values of } x$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad \text{for all values of } x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad \text{for all values of } x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots \qquad \text{for } -1 < x \le 1$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots \qquad \text{for } -1 < x < 1$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for } -1 < x < 1$$

Select Values of Trigonometric Functions:

θ	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$