# Math 116 — First Midterm — February 12, 2024

EXAM SOLUTIONS



1. This exam has 16 pages including this cover.
2. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

| Problem | Points | Score |
|---|---|---|
| 1 | 16 | |
| 2 | 15 | |
| 3 | 18 | |
| 4 | 9 | |

| Problem | Points | Score |
|---|---|---|
| 5 | 14 | |
| 6 | 11 | |
| 7 | 11 | |
| 8 | 6 | |
| Total | 100 | |

**1**. [16 points] Let $f(x)$ be a function that is **even** and **twice differentiable.** Some values of $f(x)$ and $f'(x)$ are given in the table below:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | 2 | $-1$ | 0 | 5 |
| $f'(x)$ | 0 | 4 | $\sqrt{2}$ | 1 | $e$ |

Use the table above to compute the **exact value** of the following integrals. If there is not enough information to determine the exact value of an integral, write "NEI." You need to evaluate all integrals completely, and your answers should not involve the letter $f$, but you do not need to simplify your final answers. Show all your work.

**a**. [3 points] $\displaystyle\int_{-2}^{2} f'(x)\,dx$

*Solution:* There are two possible ways to arrive at the answer:

**Solution 1** ($f(x)$ is even)**:** Since $f(x)$ is an even function, we have $f(2) = f(-2)$. So, by the First Fundamental Theorem of Calculus, $\displaystyle\int_{-2}^{2} f'(x)\,dx = f(2) - f(-2) = 0$.

**Solution 2** ($f'(x)$ is odd)**:** Since $f(x)$ is an even function, then $f(x) = f(-x)$ for all $x$. Taking derivatives of both sides and using the chain rule, $f'(x) = -f'(-x)$ for all $x$, so $f'(x)$ is an odd function. Therefore $\displaystyle\int_{-2}^{2} f'(x)\,dx = 0$ by symmetry.

**Answer:** 0

**b**. [4 points] $\displaystyle\int_{1}^{e^2} \frac{f'(\ln(t))}{t}\,dt$

*Solution:* We use a substitution $u = \ln(t)$, so that $du = \dfrac{1}{t}\,dt$. It follows that

$$\int_{1}^{e^2} \frac{f'(\ln(t))}{t}\,dt = \int_{0}^{2} f'(u)\,du = f(2) - f(0) = -1 - (-3) = 2.$$

**Answer:** 2

**c.** [4 points] $\int_1^3 (2w+1)f'(w)\,dw$

*Solution:* We integrate by parts:

$$\begin{aligned}\int_1^3 (2w+1)f'(w)\,dw &= (2w+1)f(w)\Big|_1^3 - \int_1^3 2f(w)\,dw \\ &= 7f(3) - 3f(1) - 2\int_1^3 f(w)\,dw \\ &= 0 - 6 - 2\int_1^3 f(w)\,dw.\end{aligned}$$

However, no information is given on an antiderivative of $f$, so we cannot evaluate $\int_1^3 f(w)\,dw$. Therefore the answer is NEI.

**Answer:** **NEI**

**d.** [5 points] $\int_1^2 2x^3 f''(x^2)\,dx$

*Solution:* First use a substitution $u = x^2$, so that $du = 2x\,dx$, and so we have

$$\int_1^2 2x^3 f''(x^2)\,dx = \int_1^4 uf''(u)\,du.$$

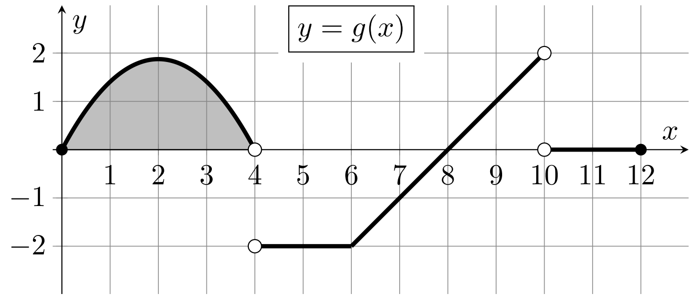
Now integrate by parts:

$$\begin{aligned}\int_1^4 uf''(u)\,du &= uf'(u)\Big|_1^4 - \int_1^4 f'(u)\,du \\ &= 4f'(4) - f'(1) - (f(4) - f(1)) \\ &= 4e - 4 - (5-2) \\ &= 4e - 7.\end{aligned}$$

**Answer:** $4e - 7$

**2**. [15 points] A function $g(x)$ is graphed below and has the following properties:

- $g(x)$ is piecewise linear for $x > 4$.
- The shaded region has area 5.



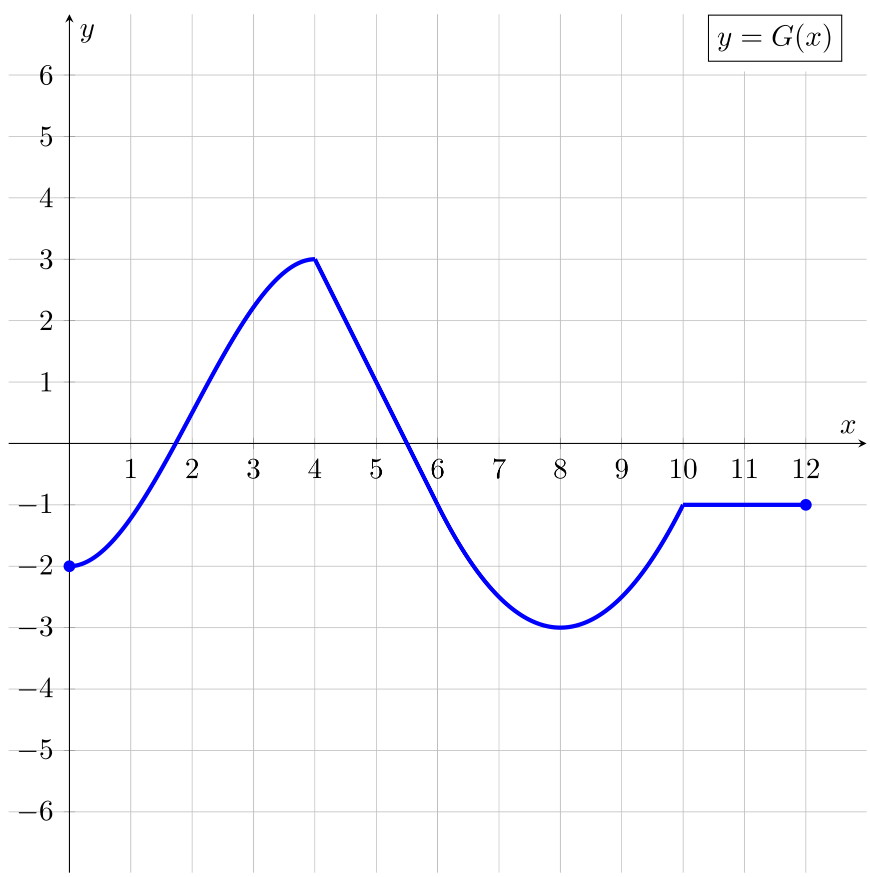

Let $G(x)$ be the continuous antiderivative of $g(x)$ satisfying $G(6) = -1$.

**a**. [5 points] Use the graph of $g(x)$ to complete the table below with the **exact** values of $G(x)$.

| $x$ | 0 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $G(x)$ | $-2$ | $3$ | $-1$ | $-3$ | $-1$ | $-1$ |

**b**. [10 points] Sketch a graph of $G(x)$ on the interval $[0, 12]$ using the axes provided below. Be sure to pay attention to:

- where $G(x)$ is and is not differentiable;
- where $G(x)$ is increasing, decreasing, or constant;
- where $G(x)$ is concave up, concave down, or linear;
- the slope of $G(x)$ at $x = 2$;
- the values of $G(x)$ you found in the table in part **a.**



*Solution:* The graph of $G(x)$ is above. Note the following:

The graph of $G(x)$ should be continuous on $[0, 12]$. Since $g(x)$ is defined and is continuous on $[0, 12]$ **except** at $x = 4$ and $x = 10$, then $G(x)$ must be differentiable on $(0, 12)$ **except** at $x = 4$ and $x = 10$.

Since $g(x)$ is positive on the intervals $(0, 4)$ and $(8, 10)$, then $G(x)$ should be increasing on $(0, 4)$ and $(8, 10)$.

Since $g(x)$ is negative on the interval $(4, 8)$, then $G(x)$ should be decreasing on $(4, 8)$.

Since $g(x) = 0$ on the interval $(10, 12)$, then $G(x)$ should be constant on $(10, 12)$.

Since $g(x)$ is increasing on the intervals $(0, 2)$ and $(6, 10)$, then $G(x)$ should be concave up on $(0, 2)$ and $(6, 10)$.

Since $g(x)$ is decreasing on the interval $(2, 4)$, then $G(x)$ should be concave down on $(2, 4)$.

Since $g(x)$ is constant on the intervals $(4, 6)$ and $(10, 12)$, then $G(x)$ should be linear on $(4, 6)$ and $(10, 12)$.

Since $g(2) \approx 2$, then the slope of $G(x)$ at $x = 2$ should be approximately 2.

Finally, the graph of $G(x)$ should contain the points $(0, -2)$, $(4, 3)$, $(6, -1)$, $(8, -3)$, $(10, -1)$, and $(12, -1)$.

**3.** [18 points] In the video game *The Legend of Helga*, the heroes Helga and Lank must save the world from the evil wizard Kanon. In the final battle, Helga and Kanon duel each other, while Lank creates magical spells to assist Helga.

**a.** [4 points] Using a magical spell costs Magic Points (MP), of which Lank has a limited amount. Let $L(t)$ be the amount of MP that Lank has remaining $t$ minutes after the battle starts. The function $L(t)$ is differentiable, and its **derivative** $L'(t)$ is given by

$$L'(t) = (1 - t^3)^{1/3}.$$

Suppose that, after 5 minutes of battle, Lank has 20 MP remaining.
Write a formula involving an integral for $L(t)$. Your answer should not involve the letter $L$.

*Solution:* By the Construction Theorem, $L(t)$ can be written in the form

$$L(t) = C + \int_a^t (1 - x^3)^{1/3}\, dx$$

for some constants $a$ and $C$. Since Lank has 20 MP remaining after 5 minutes of battle, this means that $L(5) = 20$, so we can set $a = 5$ and $C = 20$ to obtain our formula.

**Answer:** $L(t) =$ $20 + \int_5^t (1 - x^3)^{1/3}\, dx$

**b.** [5 points] While casting magical spells, Lank dodges Kanon's attacks by running along a path modeled by the curve

$$H(x) = \frac{1}{6}(4x - 1)^{3/2}$$

from $x = 1$ to $x = 16$, where distances are measured in meters.
Compute the **exact value** of the **arc length** of this curve. You need to evaluate all integrals completely, and your answer should not involve the letter $H$, but you do not need to simplify your final answer. Show all your work.

*Solution:* Recall that the arc length of $H(x)$ from $x = 1$ to $x = 16$ is given by the formula

$$\int_1^{16} \sqrt{1 + H'(x)^2}\, dx.$$

By the power rule and the chain rule, the formula for $H'(x)$ is given by

$$H'(x) = (4x - 1)^{1/2},$$

so the arc length integral is

$$\int_1^{16} \sqrt{1 + H'(x)^2}\, dx = \int_1^{16} \sqrt{1 + 4x - 1}\, dx = 2\int_1^{16} \sqrt{x}\, dx.$$

The integral can be evaluated using a standard antiderivative:

$$2\int_1^{16} \sqrt{x}\, dx = \frac{4}{3}x^{3/2}\Big|_1^{16} = \frac{4}{3}(16^{3/2} - 1^{3/2}) = 84.$$

This is our answer.

**Answer:** 84 meters

*This problem continues on the next page.*

**3. (continued)**

As Helga attacks Kanon throughout the battle, Kanon's Hit Points (HP) decrease. The amount of HP that Kanon has $t$ minutes after the battle starts is given by the differentiable function

$$K(t) = \int_{t-4}^{\sqrt{t+16}} \left(12 + \cos(w^2)\right) dw.$$

**c.** [5 points] Compute the **exact value** of $K'(8)$. You do not need to simplify your final answer.

*Solution:* First we rewrite the given function $K(t)$ as

$$K(t) = \int_0^{\sqrt{t+16}} \left(12 + \cos(w^2)\right) dw - \int_0^{t-4} \left(12 + \cos(w^2)\right) dw.$$

By the Second Fundamental Theorem of Calculus and the chain rule, we obtain

$$K'(t) = \frac{1}{2\sqrt{t+16}} \left(12 + \cos\left((\sqrt{t+16}\,)^2\right)\right) - \left(12 + \cos\left((t-4)^2\right)\right).$$

Substituting $t = 8$, we obtain our answer.

**Answer:** $K'(8) = \frac{1}{2\sqrt{24}}\left(12 + \cos(24)\right) - \left(12 + \cos(16)\right)$

**d.** [4 points] Kanon is defeated upon reaching 0 HP, at which point the battle is over. Find the **one** value of $t$ at which Kanon is defeated. Show all of your work.

*Hint: If you find multiple such values of $t$, check which ones are actually solutions.*

*Solution:* We have $K(t) = 0$ when $t - 4 = \sqrt{t+16}$, so we solve for $t$:

$$\begin{aligned} t - 4 &= \sqrt{t+16} \\ (t-4)^2 &= t + 16 \\ t^2 - 8t + 16 &= t + 16 \\ t^2 - 9t &= 0 \\ (t-9)t &= 0 \end{aligned}$$

This gives the "solutions" $t = 0$ and $t = 9$, but observe that $t = 0$ is an extraneous solution of the original equation because $0 - 4 \neq \sqrt{0+16}$. On the other hand, $t = 9$ is a solution since $9 - 4 = \sqrt{9+16}$.

**Answer:** $t = 9$

**4.** [9 points] The city of Rainneapolis has a strange weather pattern. It is always sunny, except for February 2nd, when it rains substantially all day. This year, Amin prepared for the stormy day by building a machine which continuously removes rainwater from his backyard.

- Let $T(h)$ be the **total amount** of rainwater in Amin's backyard, in cubic feet, $h$ hours after 12:00am on February 2nd.
- Let $A(h)$ be the **rate** at which the rain **adds** water to Amin's backyard, in cubic feet per hour, $h$ hours after 12:00am on February 2nd.
- Let $M(h)$ be the **rate** at which Amin's **machine removes** rainwater from his backyard, in cubic feet per hour, $h$ hours after 12:00am on February 2nd.

The functions $T(h)$, $A(h)$, and $M(h)$ are all differentiable. Assume that there is **no rainwater** in Amin's backyard **before** it starts raining at 12:00am on February 2nd.

**a.** [3 points] Which of the following gives a correct interpretation of $\int_4^{10} M(h)\,dh = 8000$? Circle **all** correct answers.

(i) The total amount of rainwater in Amin's backyard decreases by 8000 cubic feet from 4:00am to 10:00am on February 2nd.

$\boxed{\text{(ii)}}$ Between 4:00am and 10:00am on February 2nd, Amin's machine removes a total of 8000 cubic feet of rainwater from his backyard.

(iii) The rate at which Amin's machine removes rainwater from his backyard between 4:00am and 10:00am on February 2nd is 8000 cubic feet per hour.

(iv) At 10:00am on February 2nd, Amin's machine removes rainwater from his backyard at a rate of 8000 cubic feet per hour faster than at 4:00am.

(v) NONE OF THESE

*Solution:* The function $M(h)$ is the **rate** at which Amin's machine **removes** rainwater from his backyard, so its definite integral $\int_4^{10} M(h)\,dh$ gives the **amount** of rainwater that is **removed** from his backyard between 4:00am and 10:00am. Hence $\int_4^{10} M(h)\,dh = 8000$ means that the total amount of rainwater that is removed from his backyard between 4:00am and 10:00am is 8000 cubic feet, so (ii) is the answer. To explain why the other choices are incorrect:

First, (i) is incorrect because it is a statement about the **total amount** of rainwater in Amin's backyard, not the amount that is **removed.** The given equation only involves $M(h)$, which models water removal. Choice (i) describes the equation $T(10) - T(4) = -8000$, or $\int_4^{10} T'(h)\,dh = -8000$.

Next, (iii) is incorrect because it is a statement about the **rate** at which water is removed, not a statement about the **amount** of water removed. Also, (iii) suggests that this removal rate is a constant 8000 cubic feet per hour between 4:00am and 10:00am, which is not supported by the problem. Choice (iii) describes the statement "$M(h) = 8000$ for $4 \leq h \leq 10$."

Finally, (iv) is incorrect for the same reason: it describes the **rate**, not the **amount.** Choice (iv) describes the equation $M(10) - M(4) = 8000$, or $\int_4^{10} M'(h)\,dh = 8000$.

**b.** [3 points] Which of the following expressions gives the **total amount** of rainwater, in cubic feet, in Amin's backyard at 7:00am? Circle **all** correct answers.

$\boxed{\text{(i)}}$ $\displaystyle\int_0^7 T'(h)\,dh$ (ii) $\displaystyle\int_0^7 T(h)\,dh$ (iii) $\displaystyle\int_0^7 \big(A(h)+M(h)\big)\,dh$

$\boxed{\text{(iv)}}$ $\displaystyle\int_0^7 A(h)\,dh - \int_0^7 M(h)\,dh$ (v) NONE OF THESE

*Solution:* By definition of $T(h)$, the total amount of rainwater in Amin's backyard at 7:00am equals $T(7)$. So we are looking for quantities that equal $T(7)$. Note that $T'(h) = A(h) - M(h)$ by definition of rates, so by the First Fundamental Theorem of Calculus,

$$\int_0^7 A(h)\,dh - \int_0^7 M(h)\,dh = \int_0^7 (A(h) - M(h))\,dh = \int_0^7 T'(h)\,dh = T(7) - T(0) = T(7),$$

where $T(0) = 0$ because there is no rainwater in Amin's backyard before it starts raining at 12:00am on February 2nd. The above equation shows that (i) and (iv) are correct.

Now, (ii) is incorrect because $\displaystyle\int_0^7 T(h)\,dh$ is the **integral** of the total amount of rainwater, not the amount itself. Also, (iii) is incorrect because $\displaystyle\int_0^7 \big(A(h)+M(h)\big)\,dh$ is the amount of rainwater that falls in Amin's backyard **plus** the amount that is removed by 7:00am, which does not give the total amount of rainwater at 7:00am.

**c.** [3 points] Which of the following expressions gives the **average amount** of rainwater, in cubic feet, in Amin's backyard between 6:00am and 9:00am? Circle **all** correct answers.

(i) $\displaystyle\frac{1}{9-6}\int_6^9 T'(h)\,dh$ $\boxed{\text{(ii)}}$ $\displaystyle\frac{1}{9-6}\int_6^9 T(h)\,dh$ (iii) $\displaystyle\frac{T(9)-T(6)}{9-6}$

$\boxed{\text{(iv)}}$ $\displaystyle\frac{1}{3}\int_0^9 T(h)\,dh + \frac{1}{3}\int_6^0 T(h)\,dh$ (v) NONE OF THESE

*Solution:* First note that, by properties of integrals,

$$\frac{1}{3}\int_0^9 T(h)\,dh + \frac{1}{3}\int_6^0 T(h)\,dh = \frac{1}{3}\int_0^9 T(h)\,dh - \frac{1}{3}\int_0^6 T(h)\,dh = \frac{1}{3}\int_6^9 T(h)\,dh,$$

and by the First Fundamental Theorem of Calculus,

$$\frac{1}{9-6}\int_6^9 T'(h)\,dh = \frac{T(9)-T(6)}{9-6}.$$

The first equation shows that (ii) and (iv) are equivalent, and the second equation shows that (i) and (iii) are equivalent.

Now, (ii) is correct, because this is the formula for the **average value** of $T(h)$, the **amount** of rainwater, between 6:00am and 9:00am. So (iv) is also correct. On the other hand, (iii) is incorrect, as this is the **average rate of change** of the amount of rainwater from 6:00am to 9:00am, which is different from the **average value** in that time frame. So (i) is also incorrect.

**5**. [14 points]

**a**. [6 points] Split the following expression into partial fractions with two or more terms. **Do not integrate these terms.** Please clearly show all of your work.

$$\frac{5x-4}{(x-2)^2(x+1)}$$

*Solution:* The partial fraction decomposition of the given function has the form

$$\frac{5x-4}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}.$$

By multiplying to obtain a common denominator, we have

$$5x-4 = A(x-2)(x+1) + B(x+1) + C(x-2)^2. \qquad (*)$$

Below are two possible ways to complete the problem:

**Solution 1** (Comparing coefficients)**:** By multiplying out the terms on the right-hand side of the equation (*) and grouping terms with the same powers of $x$, we obtain

$$0x^2 + 5x - 4 = (A+C)x^2 + (-A+B-4C)x + (-2A+B+4C).$$

This gives the system of equations

$$A + C = 0, \qquad -A + B - 4C = 5, \qquad -2A + B + 4C = -4.$$

The first equation implies $C = -A$, so the second and third equations become

$$3A + B = 5, \qquad -6A + B = -4.$$

Taking this first equation and subtracting the second equation from it, we obtain $9A = 9$. Thus $A = 1$, so $3A + B = 5$ implies $B = 2$, and $C = -A$ implies $C = -1$. This gives us the answer.

**Solution 2** (Plugging in values)**:** By setting $x = 2$ in the equation (*), we obtain

$$5(2) - 4 = A(2-2)(2+1) + B(2+1) + C(2-2)^2.$$

This simplifies to $6 = 0 + 3B + 0$, thus $B = 2$. Now setting $x = -1$ in (*), we obtain

$$5(-1) - 4 = A(-1-2)(-1+1) + B(-1+1) + C(-1-2)^2.$$

This simplifies to $-9 = 0 + 0 + 9C$, thus $C = -1$. Finally, setting $x = 0$ in (*) and using the facts that $B = 2$ and $C = -1$, we obtain

$$5(0) - 4 = A(0-2)(0+1) + 2(0+1) + (-1)(0-2)^2.$$

This simplifies to $-4 = -2A + 2 - 4$, thus $A = 1$. This gives us the answer.

**Answer:** $\dfrac{1}{x-2} + \dfrac{2}{(x-2)^2} + \dfrac{-1}{x+1}$

**b**. [8 points] Use the partial fraction decomposition

$$\frac{4x-2}{(3-x)(x^2+1)} = \frac{1}{3-x} + \frac{x-1}{x^2+1}$$

to evaluate the following **indefinite** integral. Please clearly show all of your work.

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx.$$

*Solution:* Using the partial fraction decomposition:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x-1}{x^2+1}\,dx.$$

Now we use linearity:

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = \int \frac{1}{3-x}\,dx + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx.$$

The first integral is a standard antiderivative (or, substitute $u = 3-x$):

$$\int \frac{1}{3-x}\,dx = -\ln|3-x| + C.$$

The second integral can be evaluated using the substitution $u = x^2+1$, so that $du = 2x\,dx$, and thus

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C.$$

The third integral is a standard antiderivative:

$$-\int \frac{1}{x^2+1}\,dx = -\arctan(x) + C.$$

Putting this all together, our final answer is

$$\int \frac{4x-2}{(3-x)(x^2+1)}\,dx = -\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C.$$

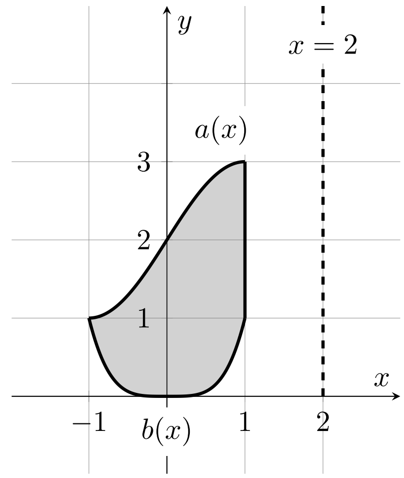
**Answer:** $-\ln|3-x| + \frac{1}{2}\ln|x^2+1| - \arctan(x) + C$

**6**. [11 points] Louise, a world-famous abstract artist and cheese enthusiast, is experimenting with new designs for cheese sculptures. She has two ideas for a cheese sculpture and would like to know the volume of each one so that she knows how much cheese to buy.

**a**. [6 points] Louise's first idea involves the shaded region to the right, which is bounded by the line $x = 1$ and the curves

$$a(x) = 2 + \sin\left(\tfrac{\pi}{2}x\right) \qquad \text{and} \qquad b(x) = x^4$$

on the interval $[-1, 1]$.




Write an integral that represents the volume of the solid formed by rotating this region around the line $x = 2$. **Do not evaluate your integral.** Your answer should not involve the letters $a$ or $b$.

*Solution:* There are two ways to solve this problem:

**Solution 1** (Shell method)**:** The shell method states that the volume of a cylindrical shell of thickness $\Delta x$ at a point $-1 \le x \le 1$ is of the form $2\pi r(x)h(x)\,\Delta x$ for some functions $r(x)$ and $h(x)$, representing the radius and height of our cylindrical shell, respectively.

For each $-1 \le x \le 1$, the distance from $x$ to the line $x = 2$ is given by $2 - x$, so $r(x) = 2 - x$ is the radius of our shell. The height of our shell is $h(x) = a(x) - b(x)$, that is, $h(x) = 2 + \sin(\frac{\pi}{2}x) - x^4$. This gives us our answer:

$$\int_{-1}^{1} 2\pi(2 - x)\left(2 + \sin\left(\frac{\pi}{2}x\right) - x^4\right) dx$$

**Solution 2** (Washer method)**:** Using the washer method is considerably more difficult: we must invert both $a(x)$ and $b(x)$, and we must split the integral at $y = 1$. Inverting these functions:

$$y = 2 + \sin\left(\frac{\pi}{2}x\right) \qquad \Rightarrow \qquad x = \frac{2}{\pi}\arcsin(y - 2),$$

$$y = x^4 \qquad \Rightarrow \qquad x = \pm y^{1/4}.$$

The washer method states that the volume of a washer of thickness $\Delta y$ at a point $0 \le y \le 3$ is of the form $\pi(R(y)^2 - r(y)^2)\,\Delta y$ for some functions $R(y)$ and $r(y)$, representing the outer radius and inner radius of our washer, respectively.

For $0 \le y \le 1$, the outer radius is $R(y) = 2 + y^{1/4}$, and the inner radius is $r(y) = 2 - y^{1/4}$.
For $1 \le y \le 3$, the outer radius is $R(y) = 2 - \frac{2}{\pi}\arcsin(y - 2)$, and the inner radius is $r(y) = 1$.

Putting this all together, this gives us our answer:

$$\int_0^1 \pi\left((2 + y^{1/4})^2 - (2 - y^{1/4})^2\right) dy + \int_1^3 \pi\left(\left(2 - \frac{2}{\pi}\arcsin(y - 2)\right)^2 - 1^2\right) dy$$

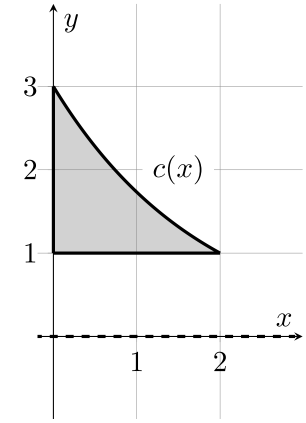
**Answer:** $\int_{-1}^{1} 2\pi(2 - x)\left(2 + \sin\left(\frac{\pi}{2}x\right) - x^4\right) dx$

**b.** [5 points] Louise's second idea involves the shaded region to the right, bounded by the curve

$$c(x) = \left(\sqrt{3}\right)^{2-x},$$

the $y$-axis, and the line $y = 1$ on the interval $[0, 2]$.

Write an integral that represents the volume of the solid formed by rotating this region around the $x$-axis.
**Do not evaluate your integral.** Your answer should not involve the letter $c$.




*Solution:* There are two ways to solve this problem:

**Solution 1** (Washer method)**:** The washer method states that the volume of a washer of thickness $\Delta x$ at a point $0 \le x \le 2$ is of the form $\pi(R(x)^2 - r(x)^2)\,\Delta x$ for some functions $R(x)$ and $r(x)$, representing the outer radius and inner radius of our washer, respectively.

For each $0 \le x \le 2$, the outer radius $R(x)$ is given by the distance from the $x$-axis to $y = c(x)$, which is $R(x) = (\sqrt{3})^{2-x}$. The inner radius $r(x)$ is given by the distance from the $x$-axis to $y = 1$, which is $r(x) = 1$. This gives us our answer:

$$\int_0^2 \pi\left(\left((\sqrt{3})^{2-x}\right)^2 - 1^2\right) dx$$

**Solution 2** (Shell method)**:** The shell method requires an additional step, but here it is not too complicated. First we invert the function $c(x)$:

$$y = (\sqrt{3})^{2-x} \qquad \Rightarrow \qquad x = 2 - \frac{\ln(y)}{\ln(\sqrt{3})}.$$

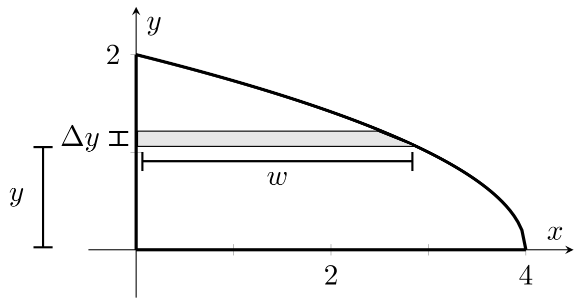
The shell method states that the volume of a cylindrical shell of thickness $\Delta y$ at a point $1 \le y \le 3$ is of the form $2\pi r(y)h(y)\,\Delta y$ for some functions $r(y)$ and $h(y)$, representing the radius and height of our cylindrical shell, respectively.

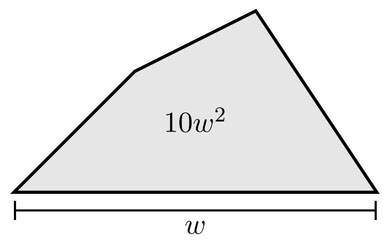
For each $1 \le y \le 3$, the radius of our shell is $r(y) = y$, and the height of our shell is $h(y) = 2 - \frac{\ln(y)}{\ln(\sqrt{3})}$. This gives us our answer:

$$\int_1^3 2\pi y\left(2 - \frac{\ln(y)}{\ln(\sqrt{3})}\right) dy$$

**Answer:** $\int_0^2 \pi\left(\left((\sqrt{3})^{2-x}\right)^2 - 1^2\right) dx$

**7.** [11 points] In an accidental discovery, scientists created the Ultra Bouncy Toy (UBT), which bounces unpredictably due to its unusual shape and irregular density.
The base of the UBT is the region bounded by $y = \sqrt{4-x}$, the $x$-axis, and the $y$-axis, shown below to the left. All distances are measured in centimeters (cm). A sample slice of the base of width $w$ and thickness $\Delta y$ is shown in the graph below to the left. Cross-sections of the UBT perpendicular to the ***y*-axis** have the shape shown below to the right. The area of such a cross-section is $10w^2$.




**a.** [3 points] Write a formula in terms of $y$ for the width $w$ of a slice that is $y$ centimeters above the $x$-axis. **Include units.**

*Solution:* We are given $y = \sqrt{4-w}$. Solving for $w$, we have $w = 4 - y^2$.

**Answer:** $w =$ $4 - y^2$ **Units:** **cm**

**b.** [3 points] Write an expression that approximates the **volume of a slice** of the UBT that is $y$ centimeters above the $x$-axis and has thickness $\Delta y$ centimeters. Your answer should not involve the letter $w$. **Include units.**

*Solution:* The approximate volume of a slice is given by (volume) = (area of cross-section) $\Delta y$. Using part (a), the area of a cross-section is $10w^2 = 10(4-y^2)^2$. Thus the volume of a slice is about $10(4-y^2)^2\,\Delta y$ cm$^3$.

**Answer:** $10(4-y^2)^2\,\Delta y$ **Units:** **cm$^3$**

The density of the UBT is given by the function $\delta(y)$, measured in grams per cubic centimeter (g/cm$^3$), where $y$ is the distance from the $x$-axis in centimeters.

**c.** [2 points] Write an expression that approximates the **mass of a slice** of the UBT that is $y$ centimeters above the $x$-axis and has thickness $\Delta y$ centimeters. Your answer may include $\delta$, but it should not involve the letter $w$. **Include units.**

*Solution:* The approximate mass of a slice is given by (mass) = (volume)(density). At a distance $y$, the density in an entire slice is approximately $\delta(y)$ g/cm$^3$. By part (b), the volume of a slice is $10(4-y^2)^2\,\Delta y$ cm$^3$. Thus the mass of a slice is about $10(4-y^2)^2\delta(y)\,\Delta y$ g.

**Answer:** $10(4-y^2)^2\delta(y)\,\Delta y$ **Units:** **g**

**d.** [3 points] Write an expression involving an integral that represents the **total mass** of the UBT. Your answer may include $\delta$. **Include units.**

*Solution:* By part (c), the mass of a slice is approximately $10(4-y^2)^2\delta(y)\,\Delta y$. The total mass is obtained by taking the integral of the mass of all slices as $\Delta y \to 0$ for $0 \le y \le 2$. This gives our answer in the form of an integral.

**Answer:** $\displaystyle\int_0^2 10(4-y^2)^2\delta(y)\,dy$ **Units:** **g**

**8.** [6 points] Each part below describes a twice differentiable function and one or more approximations of its integral. For each of the following statements, determine if the statement is **ALWAYS** true, **SOMETIMES** true, or **NEVER** true, and circle the appropriate answer.
No justification is required.

**a.** [1 point] If $A'(x) > 0$ for all $x$, then $\text{LEFT}(4) \leq \int_{-1}^{1} A(x)\,dx$.

*Circle one:* $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

*Solution:* Since $A'(x) > 0$, then $A(x)$ is always increasing on $[-1, 1]$, so LEFT(4) is an underestimate of $\int_{-1}^{1} A(x)\,dx$.

**b.** [1 point] If $B'(x) > 0$ for all $x$, then $\text{TRAP}(4) \leq \int_{-1}^{1} B(x)\,dx$.

*Circle one:* **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

*Solution:* Since $B'(x) > 0$, then $B(x)$ is always increasing on $[-1, 1]$. But we are not given whether $B(x)$ is concave up or concave down on $[-1, 1]$, so TRAP(4) could be an underestimate (for example, if $B(x) = -e^{-x}$) or an overestimate (for example, if $B(x) = e^{x}$) of $\int_{-1}^{1} B(x)\,dx$.

**c.** [1 point] If $C''(x) > 0$ for all $x$, then $\text{TRAP}(4) \leq \int_{-1}^{1} C(x)\,dx$.

*Circle one:* **ALWAYS** **SOMETIMES** $\boxed{\textbf{NEVER}}$

*Solution:* Since $C''(x) > 0$, then $C(x)$ is always concave up on $[-1, 1]$, so TRAP(4) is an overestimate of $\int_{-1}^{1} C(x)\,dx$, not an underestimate. Moreover, since $C''(x) > 0$, then $C(x)$ has no inflection points, so we cannot even have $\text{TRAP}(4) = \int_{-1}^{1} C(x)\,dx$.

**d.** [1 point] If $D(x)$ is odd and MID(4) approximates $\int_{-1}^{1} D(x)\,dx$, then $\text{MID}(4) = 0$.

*Circle one:* $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

*Solution:* By direct computation, $\text{MID}(4) = D(-\frac{3}{4}) + D(-\frac{1}{4}) + D(\frac{1}{4}) + D(\frac{3}{4})$. But $D(x)$ is odd, so $D(-\frac{3}{4}) = -D(\frac{3}{4})$ and $D(-\frac{1}{4}) = -D(\frac{1}{4})$, therefore $\text{MID}(4) = 0$.

**e.** [1 point] If $E'(x) > 0$ and $E''(x) < 0$ for all $x$, then $\int_{-1}^{1} E(x)\,dx \le \text{MID}(2) \le \text{RIGHT}(2)$.

*Circle one:* $\boxed{\textbf{ALWAYS}}$ **SOMETIMES** **NEVER**

*Solution:* Since $E'(x) > 0$ and $E''(x) < 0$, then $E(x)$ is always increasing and concave down on $[-1, 1]$, so both RIGHT(2) and MID(2) are overestimates of $\int_{-1}^{1} E(x)\,dx$. Moreover, $E(x)$ always increasing implies that MID(2) is a better approximation of the integral than RIGHT(2). More specifically, we note that

$$\text{MID}(2) = E(-\tfrac{1}{2}) + E(\tfrac{1}{2}) \qquad \text{and} \qquad \text{RIGHT}(2) = E(0) + E(1),$$

and since $E(x)$ is always increasing, then $E(-\frac{1}{2}) \le E(0)$ and $E(\frac{1}{2}) \le E(1)$, so therefore

$$\text{MID}(2) = E(-\tfrac{1}{2}) + E(\tfrac{1}{2}) \le E(0) + E(1) = \text{RIGHT}(2).$$

**f.** [1 point] If $F(x)$ is not constant, then RIGHT(3) approximates the integral $\int_{-1}^{1} F(x)\,dx$ more accurately than RIGHT(2).

*Circle one:* **ALWAYS** $\boxed{\textbf{SOMETIMES}}$ **NEVER**

*Solution:* If $F(x) = x$, then RIGHT(2) $= 1$ and RIGHT(3) $= \dfrac{2}{3}$, while $\int_{-1}^{1} F(x)\,dx = 0$, so RIGHT(3) is a more accurate estimate than RIGHT(2). On the other hand, consider the function

$$F(x) = \begin{cases} 1 & x \le 0, \\ 1 - 6(3x)^5 + 15(3x)^4 - 10(3x)^3 & 0 < x < \frac{1}{3}, \\ 0 & x \ge \frac{1}{3}. \end{cases}$$

(Draw a picture!) By direct computation, RIGHT(2) $= 1$ and RIGHT(3) $= \dfrac{2}{3}$, but $\int_{-1}^{1} F(x)\,dx > 1$, so RIGHT(2) is a more accurate estimate than RIGHT(3) in this case. (Note that this shows the statement can be false even if $F(x)$ never increases.)